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Thos Gibson

A
TREATISE
ON THE
RECTILINEAR MOTION
AND
ROTATION OF BODIES;
WITH A
DESCRIPTION
OF
ORIGINAL EXPERIMENTS
RELATIVE TO THE SUBJECT.

By G. ATWOOD, M. A. F. R. S.

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TO HIS GRACE

THE DUKE OF RUTLAND,

KNIGHT OF THE MOST NOBLE ORDER OF THE GARTER,

LORD LIEUTENANT OF IRELAND,

&c. &c. &c.

THE FOLLOWING PAGES

ARE WITH GREAT RESPECT

AND GRATITUDE

INSCRIBED

BY HIS GRACE'S

MOST OBEDIENT,

AND MOST OBLIGED,

HUMBLE SERVANT,

GEORGE ATWOOD.

Trinity College, Cambridge,
18 April, 1784.

PREFACE.

THE principles of rectilinear motion and rotation are of conſiderable extent in the theory of mechanics, and comprize moſt caſes in which this ſcience can be applied to practical uſe; it is probably from the facility and exactneſs with which theſe kinds of motion are produced in bodies, and are communicated from one body to another, that they have been ſo generally adopted in mechanic conſtructions of every kind.

A few curſory remarks on the enſuing ſections, may be ſufficient to explain the plan and deſign of this treatiſe.

The firſt and ſecond ſections contain the elementary propoſitions on which the theory of mechanics is founded. In the third ſection the rectilinear motion of bodies, impelled or reſiſted by forces which act uniformly, is conſidered. The depths to which ſpherical bodies, impinging with given velocities, ſink into banks of earth, ſolid blocks of timber, clay, ſnow, &c. are determined by propoſitions inſerted in the third ſection, becauſe the forces by which thoſe ſubſtances reſiſt the penetration of ſpherical bodies are, in a phyſical ſenſe, uniform: in ſtrictneſs, the forces of reſiſtance are uniform only after the ſpheres have ſunk to a depth equal to their radii; and the greater proportion the ſpace to which any ſphere penetrates, before its motion is deſtroyed, bears to its radius, the leſs error will be occaſioned by aſſuming the reſiſtance as altogether uniform. If the velocity of impact is produced by the acceleration of gravity, the action of the ſphere's weight, after it has ſunk to a depth equal to its radius, is too ſmall to have ſenſible effect in experiments made on the reſiſting forces of ſubſtances, which are in general much greater than the force of gravity.

The notation of ratios explained in the ſecond, and applied in the third ſection, is ſomewhat different from that which is commonly uſed: each ratio is repreſented by a fraction, the numerator of which is the antecedent, and denominator the conſequent of the ratio. Although the meaning and value of any ratio is no ways dependent on the manner of expreſſing it, yet if one method of notation ſhould appear leſs liable to ambiguity, and better adapted for the purpoſe of applying and explaining the theory than another, there ſeems ſufficient reaſon for a preference.

The fourth ſection contains propoſitions which determine the motion of bodies, produced by forces varying in ſome ratio of the diſtances from a fixed point; by claſſing propoſitions according to the particular law in which the accelerating force varies, ſeveral phenomena, apparently quite unconnected, admit of ſolution from the ſame principle: thus, the vibration of a pendulum in a cycloidal arc, the undulatory motion of fluids in the canal deſcribed in page (104)*, the ſmall vibrations of an elaſtic ſtring, conſidering the whole maſs to be concentrated in the middle point, are all explained from the problem for determining the motion of bodies, which are impelled by forces varying in the direct ratio of the diſtances from a fixed point.

The propoſitions juſt mentioned might

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* N.B. In the firſt line of the note inſtead of *AFIKCM* it ſhould be read *EFIKCD*.

have been inferred from this problem as corollaries, but they have been ſeparately conſidered on account of the many conſequences which follow from each. The imaginary caſe of the elaſtic ſtring naturally ſuggeſted ſome further conſideration of the ſubject, on principles more nearly correſponding to the real vibration of an elaſtic chord.

The theory of reſiſting forces, which vary in the direct duplicate ratio of the velocities, is conſidered in the fifth ſection, and is applied to the explanation of various phenomena: this is the caſe that really obtains, when bodies move in a fluid, the parts of which are free from coheſion and tenacity, provided the compreſſion of the fluid is ſuch as will cauſe the ſpace deſerted by the moving body during its paſſage, to be filled up immediately by the ſurrounding fluid.

Experiments on the reſiſtances of fluids to ſolid bodies moving through them, have not been very numerous: thoſe which are deſcribed in the 2d Vol. of the Principia, are ſufficiently deciſive to aſcertain the agreement between the author's theory and matter of fact; but the repetitions of experiments, by which any important truths are verified, cannot be thought ſuperfluous: we cannot make too frequent appeals to experience; theories, however perfect, are never ſo ſatisfactory, as when they are illuſtrated by repeated and accurate trials.

The experiments on the deſcent of ſpherical bodies in water, inſerted in ſect. v. were conſtructed with great attention to exactneſs, and the reſults were noted before the computations were made.

In pages (156) and (161) the velocities with which the aerial bubbles or ſpherules aſcend in fluids are eſtimated, and the theory is applied to the deduction of various conſequences: according to Dr. Halley's hypotheſis, vapour or ſteam conſiſts of hollow ſpherules, filled with an elaſtic fluid ſo rare, that the weight of a hollow globule and its contents, ſhall be lighter than an equal bulk of the air which ſurrounds it. The aſcent of theſe ſpherules is ſimilar to that of the air bubbles juſt referred to, which riſe in water and other fluids; and the velocities of aſcent in both caſes muſt be eſtimated from the ſame principles. A few thoughts which preſented themſelves on conſidering the ſubject in this manner, are contained in page (172,&c.). The hypotheſis, which aſcribes the production of vapour to chemical ſolution, may, perhaps, be ſufficient to account for the abſorption of water in air, but will not ſo well explain the aſcent of ſteam under all its various appearances. The power by which ſteam riſes from boiling water, or from bodies ſubjected to degrees of heat in chemical diſtillation, ſeems to be different from any that can be attributed to the ſolution of one fluid in another. Indeed, when metallic or other bodies, are diſſolved in their proper menſtrua, the ſmall aerial bubbles, which conſtantly aſcend during the ſolution, carry up particles of the bodies that are diſſolved, and in ſome degree diffuſe them over the whole maſs of the ſolvent: but no ſuch bubbles are ſuppoſed to exiſt during the aſcent of ſteam according to the hypotheſis which explains that phenomenon by referring it to the effects of chemical ſolution.

Solid particles of matter of any ſhape, deſcend in fluids according to laws correſponding in ſome degree to thoſe which regulate the deſcent of ſpherical particles: for example, large particles of any ſhape will deſcend faſter than thoſe which are much ſmaller, if they are of the ſame denſity.

According to Dr. Woodward's theory of the deluge, when mineral and other ſubſtances were by the violence of the waters, or other cauſes, diſſolved into a maſs conſiſting of heterogeneous particles of ſolid bodies ſuſpended in water, it is ſuppoſed that the denſeſt bodies would ſubſide with the greateſt velocities, and conſequently, that the ſtrata under the earth's ſurface, muſt be diſpoſed according to the order of their ſpecific gravities; but from prop. x. of the fifth ſection, it appears, that the velocities with which ſolid particles immerſed in a fluid deſcend by gradually ſubſiding, depend on their magnitudes as well as on their denſities; conſequently if the texture of any ſtratum of earth ſhould be ſuch as diſpoſes it to be divided or broken into larger atoms, than thoſe of which the lighter ſtratum conſiſts, and their magnitudes are in a due proportion, the lighter particles will ſubſide below thoſe which are ſpecifically heavier.

Strata of foſſil and other earths are found, by accurate examination, to be diſpoſed in no regular order in reſpect of their ſpecific gravities, but from what has been advanced concerning the manner in which the atoms or particles of matter ſubſide in a fluid, thoſe objections againſt Dr. Woodward's hypotheſis, which are grounded on the promiſcuous diſpoſition of the ſtrata, may appear to have leſs weight.

The principles of rotation are demonſtrated in the ſixth ſection. This theory is applied to explain the motion of pendulums which vibrate in circular arcs; to eſtimate the effects produced by the mechanic powers, or combinations of them, and to the ſolution of various problems.

In books of mechanics many experiments have been deſcribed by which the equilibrium of the mechanic powers, the compoſition and reſolution of forces, and other ſtatical principles are explained and verified; but no account is to be found of methods by which the principles of motion may be ſubjected to deciſive and ſatisfactory trials. An attempt has been made to ſupply this deficiency in the ſeventh and eighth ſections of this treatiſe. The ſeventh ſection contains the deſcription of experiments on the rectilinear motion of bodies, both accelerated and retarded; and the experiments on the principles of rotation, including thoſe which relate to the vibrations of pendulums, are inſerted in the eighth ſection. Moſt of the fundamental properties of rectilinear motion and rotation, are illuſtrated by experiments deſcribed in theſe two ſections. The numbers ſet down were the reſults of meaſurements and computations made with great care and attention: in order to inſure the reſults true to one or two decimal places, the quantities are in ſome inſtances expreſſed to a greater number of places than may ſeem neceſſary. Theſe experiments and the explanations of them, were part of a courſe of experimental lectures on the principles of natural philoſophy, read in the univerſity of Cambridge.

The hypotheſes which aſcribe permanent quantities of motion to bodies moving with given velocities, ſeem to have been adopted for the purpoſe of avoiding the difficulties which occur in ſolving moſt caſes in practical mechanics; for if the effects of forces could be truly eſtimated by a meaſure, conſiſting of the quantity of matter moved and any power of the velocities,

cities, there could be no occaſion to conſider the variation of the forces of acceleration or reſiſtance, ſince the ultimate effects produced would be known from the due application of the hypotheſes without further inveſtigation.

The chief intention of the ninth ſection is to examine into theſe hypotheſes, and to ſhew that they ought not be admitted in mechanics as general principles; many inſtances of error might be produced, which have been occaſioned by adopting them; one or two will be ſufficient to juſtify what has been aſſerted.

In Emerſon's Fluxions, p. 177, there is this problem: The radii of a wheel and axle are given in the proportion of $b : a$; a weight w acting by means of a line on the circumference of the wheel, elevates a weight y ſuſpended from a line, which goes round the axle; it is required to aſſign the quantity y, when $y \times$ into its velocity generated in a given time, is the greateſt poſſible.

In the ſolution the author ſuppoſes the momentum of bodies to be as the quantity of matter into the velocity generated; and according to the uſual doctrine of momentum, aſſumes it as an univerſal truth, that if a force acts on any different quantities of matter for a given time, it will always generate the ſame moment, eſtimated by the quantity of matter into the velocity. From this reaſoning he deduces the weight ſought $y = \sqrt{2-1} \times \frac{bw}{a}$, whereas its true value is $y = w \times \sqrt{\frac{b^4}{a^4} + \frac{b^3}{a^3} - \frac{b^2}{a^2}}$ (page 249) agreeing with the former only in the extreme caſe, when $b = a$, that is, when the radius of the wheel is equal to that of the axle.

Dr. Deſaguliers in his courſe of experimental philoſophy, vol. i. p. 173, applies the doctrine of momentum to inveſtigate the preſſure ſuſtained by the axis of a wheel and axle, when a weight p acting at the diſtance b, elevates a weight q applied at the diſtance a from the axis.

He eſtimates the momentum of bodies by the quantity of matter into the velocity generated in a given time; and lays it down as a general rule, that " the momentum produced is always equal to the momentum which produces it:" by reaſoning from this principle he deduces the preſſure on the axis, from the action of the two weights to be $= q + \frac{3pqba - qqaa}{pbb + qab}$.

By another ſolution he finds the preſſure on the axis $= \frac{qp \times \overline{b+a}^2}{pb^2 + qa^2}$; wholly different from the former, and yet both the reſults are conſequences from the ſame data: nor will it be poſſible from any reaſoning the author uſes in theſe ſolutions, to diſtinguiſh which of them is true, or to determine whether both are not erroneous; although from the general principles of motion, independent of any conſideration of momentum, it is without difficulty inferred, that the true preſſure on the axis is the latter value $\frac{qp \times \overline{b+a}^2}{pb^2 + qa^2}$.

Many other examples of a ſimilar kind might be produced; theſe two have been mentioned, not with a view of cavilling at writers who have deſerved ſo well of philoſophy, but to ſhew how much prejudice muſt be occaſioned to ſcience by aſſuming hypotheſes of momentum as general principles, when even the moſt experienced perſons have been led into error by them. In the ſections preceding the ninth of this treatiſe, the reader will find the hypotheſis of momentum uſed in explaining ſome particular caſes in mechanics; but it is in no inſtance aſſumed as a principle from which conſequences are inferred in the ſolution of problems.

In the tenth ſection the principles of rotation in free ſpace are deduced from thoſe which are demonſtrated in the ſixth ſection concerning the rotation of bodies round fixed axes.

Analytical demonſtrations have been adopted, as being moſt conſiſtent with the general plan of the work. If this method of arriving at truth appears leſs eligible in ſome points of view, in others it certainly has its advantages; particularly in being eaſily applied to the inveſtigation of problems, and in being comprized in a ſmaller compaſs than would in general be neceſſary for treating the ſame ſubject geometrically.

The enſuing treatiſe, however, is not intended to precede the ſtudy of thoſe authors who have written geometrically on the principles of motion, but is rather to be conſidered as auxiliary and ſubſervient to them. It is generally allowed that no ſpecies of mathematical reaſoning contributes ſo much to improve and ſtrengthen the mind, as that which has for its objects the properties of ſpace, whether they are purely abſtract, or ſuch as are joined with the explanation of natural phenomena: but whoever wiſhes to extend the uſe of the mathematical ſciences ſtill further, by applying them to the inveſtigation of new abſtract truths, or to the ſolution of ſuch phyſical problems as any occaſion may preſent, will ſoon perceive the neceſſity of not confining himſelf to the ſtudy of geometry only. It has been often remarked, that thoſe who perfectly underſtand the geometrical propoſitions, which explain any branch of philoſophy, without having conſidered the ſubject under an analytical form, find great difficulty in applying their theory to practice, even in the moſt ordinary caſes.

The following pages are preſented to the public as an imperfect ſketch, rather than as a work finiſhed according to the wiſh and ſatisfaction of the Author, who began this treatiſe with expectations of continuing in an academical eſtabliſhment, which would have afforded him means and opportunities of rendering it more worthy of the public eye. The book, however, ſuch as it is, has coſt him ſome trouble; which he will think amply repaid, if it ſhould appear in any degree to merit the approbation of philoſophical and mathematical readers.

The Author takes this opportunity of publicly expreſſing his thanks and acknowledgements for the liberal aſſiſtance he has received toward the expences of printing this volume, from the Vicechancellor and Syndics of the Univerſity Preſs.

Trinity College, Cambridge,
Apr. 18, 1784.

Theſe and innumerable other forces which originate from cauſes however different, are yet referred to the ſame general principles of motion, which it is the buſineſs of Mechanics to explain.

Among the various laws obſerved during the acceleration and retardation of bodies, to which every material ſubſtance in nature is ſubject as far as human obſervation extends, three, called Laws of Motion, are aſſumed as Phyſical Axioms; being propoſitions which although the mind does not aſſent to on intuition, yet as they are of the moſt obvious and intelligible kind, ſuggeſted conſtantly by the ordinary motion and quieſcence of bodies, and confirmed by every experiment which can be made on the operation of forces, as well as by ſuch * arguments as the nature of the ſubject will admit of, appear the moſt proper to be received as principles from which the theory of motion in general may be regularly deduced.

* Vid. Sect. IX. p. 358.

II.

Axiom 1. Every body perſeveres in its ſtate of reſt or uniform motion in a right line, until a change is effected by the agency of ſome external force.

2. Any change effected in the quieſcence or motion of a body is in the direction of the force impreſſed, and is proportional to it in quantity.

3. Action and reaction are equal and in contrary directions.

In comparing the changes cauſed in the motion of bodies, according to the ſecond law of motion, the times of effecting ſuch changes are underſtood to be equal: Vid. Newt. Princip. Cor. 1. to the axioms in which this ſecond law of motion is referred to.

1. That a body once at reſt will continue ſo until it is acted on by ſome external force is a truth ſo obvious that it needs no comment. Alſo a moving body having no

more power to alter the direction or velocity of its motion than to begin at firſt to move, muſt neceſſarily continue to proceed uniformly in a ſtraight line, till a change is effected by the agency of ſome external force.

2. It is likewiſe rational to ſuppoſe that any change, whether in the direction or velocity of a body's motion, ſhould be proportional to the cauſes producing ſuch change, that is, to the forces impreſſed, the time of their action being the ſame.—Thus imagine a body to be moving from A to B, Fig. 1. if during the time of its motion, a force be impreſſed in the direction of AC or BD, which is parallel to it, this impreſſed force will deflect the body from its original direction, but will not accelerate or retard the time of its arrival at the line BE; it will however cauſe [illegible] the diſtances BD, BE at which it arrives in that line, to become greater or leſs, according to the magnitude of the force which acts in the direction BE: for the ſame reaſon, if during a body's motion in the direction AC, a force in the direction AB be impreſſed, the time of the body's arrival at the line CF which is parallel to AB will not be altered, only the diſtances CD, CF in that line, will vary proportionally to the force acting in the direction CF. If therefore the two forces which act upon a body at A for a given time, be ſuch as when ſingly applied would cauſe the body to deſcribe the lines AC, AB reſpectively in the ſame time, theſe forces being applied together, the body by their joint action muſt at the end of the time have arrived at both the lines CF and BE, that is, at their common interſection D, in the extremity of the diagonal of the parallelogram $ABCD$, the two ſides of which are in the direction of the two forces, and are in quantity equal to the ſpaces which would be deſcribed by the forces acting ſingly for the ſame time.

3. When a body deſcribes equal ſpaces in equal times, the motion is ſaid to be uniform. The degree or celerity of motion is merely relative, and implies the compariſon of the ſpaces deſcribed by bodies moving uniformly for the ſame time: the proportion of the velocities will be the ſame as that of the ſpaces. The quantity of motion generated in a body, is the effect whereof the moving force is the cauſe. Quantities of motion therefore communicated to bodies in the ſame time are proportional to the moving forces.

Moving bodies alſo become ſubſequently the cauſe of motion in other bodies by impinging on them, but ſince matter poſſeſſes not in itſelf a power of creating or deſtroying the leaſt motion, it follows that as much as is gained by

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the body ſtruck, will be loſt by the other body, ſo that the quantity of motion exiſting in bodies, will be the ſame before and after the impact. From the ſame principle it follows, that a body any how attracting another body is attracted by it with equal force, and it appears from every experiment which can be made on bodies which approach each other in a right line, by means of their mutual attraction, that they move with velocities which are inverſely proportional to the quantities of matter contained in them: It would follow from hence immediately, that the quantities of motion in moving bodies, are proportional to their quantities of matter and velocities jointly; the ſame concluſion however may be deduced from the laws of motion only, independent of any experiments.

Theſe three phyſical propoſitions having been aſſumed as principles of motion, reduce the ſcience of mechanics to mathematical certainty, ariſing not only from the ſtrict coherence of innumerable properties of motion deduced from them a priori, but from their agreement with matter of fact, which agreement is ever ſeen moſt conſpicuous when moſt ſeverely and minutely examined: It is from theſe conſiderations that the laws of motion have been eſteemed not only phyſically but mathematically true.

III.

Any force acting upon a body continually in the ſame direction, will produce a continual acceleration or retardation of its motion.

Any velocity* produced in a body will be continued uniform, after the action of the force ceaſes; but as long as the force acts, new velocity† muſt be generated at every inſtant of the body's motion. Thus a body deſcending to the earth by the force of gravity, deſcribes unequal ſpaces in equal times, its motion being continually accelerated. Alſo the elaſtic ſteam of gunpowder acting upon a muſket ball, cauſes a perpetual acceleration of motion during its progreſs through the barrel; whereas if the ball impinging againſt a block of wood penetrates into its ſubſtance, the retarding force of the block cauſes a continual diminution of the initial velocity of impact, till the whole motion of the ball is deſtroyed.

* 1ſt. Law of motion.

† 2d. Law of motion.

The laws of acceleration and retardation will depend on thoſe of the force, which may be infinitely varied. Theſe different forces divide themſelves into two ſorts, viz. variable and conſtant, which diſtinction ſhould be next deſcribed.

IV.

If during a body's motion, equal velocities be communicated or deſtroyed in equal ſucceſſive portions of time, the force is ſaid to be conſtant, and the velocity uniformly accelerated or retarded.

By the ſecond law of motion, any change produced in the velocity of a moving body, is in the ſame direction with the force impreſſed, and is proportional to it in quantity, the time being given: wherefore by the propoſition, equal velocities being produced or deſtroyed in equal times, it follows that the force acts equably.

Of this ſort is the force of gravity, which in the deſcent or aſcent of a body creates or deſtroys a velocity of $32 \frac{2}{12}$ feet in each ſecond of time; alſo the force whereby a block of wood reſiſts the progreſs of a ball entering into it, deſtroys equal velocities in equal times.

V.

When unequal velocities are generated or deſtroyed in equal ſucceſſive portions of time, the force is ſaid to be variable.

By the ſecond law of motion the force is as the velocity generated or deſtroyed in the ſame body in a given time, which velocity, by the propoſition, is variable: It follows therefore, that the force is variable.

When a body proceeds with accelerated or retarded motion, the velocities in any points of the ſpace deſcribed, are eſtimated by finding what ſpaces would be deſcribed by the velocities acquired in thoſe points, and continued uniform for a given time: the ſpaces deſcribed will give the ratio of the velocities required.

In like manner to eſtimate a variable force acting upon a body in any two points of the ſpace deſcribed, it is neceſſary to find what would be the velocities generated or deſtroyed in a given time, by the force acting in thoſe points, and continued conſtant; the ratio of theſe velocities will give the ratio of the forces required.

It appears therefore that the properties of variable forces muſt be referred to thoſe of conſtant forces: theſe are conſidered in the enſuing propoſitions.

Although the properties of acceleration and thoſe of retardation are equally deducible from the laws of motion, and might be demonſtrated together, yet it will be more convenient to conſider them ſeparately. The laws of acceleration being demonſtrated, thoſe of retarded motions will follow from them.

VI.

When a body is acted upon by a conſtant force, there are four quantities which become the objects of mechanical conſideration, viz. the ſpace deſcribed, the time of deſcription, the velocity acquired, and the force which produces it; any two of which being given, the others may be aſcertained.

It may be remarked, that the force here mentioned, relates to the communication of velocity only, any difference in the quantities of matter moved, not being conſidered: It is called the accelerating force, being proportional to the velocity generated in a given time.

The moving force relates to the quantity of matter moved, as well as the velocity communicated, and it is proportional to the quantity of motion produced in a given time.

The diſtinction between theſe forces will further appear from conſidering what is the meaſure of the quantities of motion communicated to bodies, or of the moving forces by which ſuch motion has been generated.

VII.

The moving forces which communicate the ſame velocity in a given time, to different bodies, will be as the quantities of matter contained in the bodies moved.

For if one body contain a quantity of matter ten times greater than another, then may the heavier be divided into ten bodies, each equal in quantity of matter to the lighter; and whatever force be required to produce a certain velocity in the lighter body, ten of theſe forces will be neceſſary to impel the ten bodies through the ſame ſpace in the ſame time, ſo that the velocities of all the bodies ſhall be equal at the end of the motion; and it is the ſame as to the velocity produced, whether the bodies be ſeparated or united, the ten forces ſtill acting upon them.

Thus the moving force exerted by gravity upon bodies deſcending towards the earth's centre, is proportional to the quantities of matter contained in them; for it appears from experiment, that all bodies, whatever be their weights deſcend near the earth's ſurface through equal ſpaces in equal times, acquiring the ſame velocity in their deſcent, the air's reſiſtance being removed, and it is manifeſt from the propoſition, that the heavier bodies will require a greater force to move them than the lighter, according to the quantities of matter contained in them.

VIII.

The moving forces acting upon bodies and the quantities of motion communicated to them in a given time, are proportional to the quantities of matter moved and the velocities communicated jointly.

For when the velocity communicated in a given time is the ſame, the moving force * is as the quantity of matter moved; and † when the quantity of matter is given, the mov-

* Art VII.

† 2d. Law of motion.

moving force is as the velocity communicated in the ſame time; therefore both quantities of matter and velocities communicated in the ſame time, being different, the moving forces, and their effects the quantities of motion produced, will be as the quantities of matter and velocities communicated jointly.

It is ſometimes aſſerted, that there are two methods whereby motion may be communicated or deſtroyed, viz. either by the continual action of a moving force, or by inſtantaneous impact; but the latter way can obtain only in perfectly hard and inflexible bodies, which exiſt not in nature; and even in the abſtract conſideration of theſe, as well as of other caſes in mechanicks, when metaphyſical poſſibilities inſtead of the natural ſtate of bodies are attended to, difficulties ariſe hardly explicable by any method of reaſoning: but it is certain, that when finite velocity is communicated to any natural body, the time wherein it is communicated muſt be finite, ſo that when the body acted upon begins to move from quieſcence, it will during the action of the force poſſeſs all the intermediate degrees of velocity, between 0 and the velocity ultimately communicated.

To exemplify this further, let us imagine that a ſoft and flexible ſubſtance, ſuch as a ball of clay impinges againſt another ſubſtance of the ſame ſort, in the direction of a line joining the centres of the balls. At the firſt inſtant of the impact, the body ſtruck will begin to move, and will proceed with a velocity inferior to that of the impinging body, the velocity of which will continue to decreaſe, and that of the other body to increaſe as long as the impact cauſes a change in the figure of the two bodies, that is, till they ſhall have poſſeſſed a velocity common to both, at which inſtant all acceleration ceaſes; provided the bodies be perfectly nonelaſtic. If the bodies be of ſuch a kind as after having received impreſſion from any impact poſſeſs a power of reſtoring their changed figure with a force equal to that of the impact, it is manifeſt, that whatever velocity was communicated during the change of figure, an equal velocity will be ſuperadded during the reſtoration of it. In this caſe after the acceleration ariſing from the impact during the change of the bodies figure has ceaſed, the bodies having then acquired a common velocity, a new acceleration will begin, being cauſed by the elaſtic force of the balls, which acting in a direction of the lines joining their centers, tends to ſeparate them, accelerating the ball ſtruck and retarding the other.

From theſe conſiderations it appears, that in whatever degree the hardneſs of perfectly elaſtic bodies may differ, the effects of their impacts on each other will be the ſame, the weights and velocities before the ſtroke being given. For the figures of the ſtriking and of the other body muſt continually change, till they have acquired a common velocity, which depends only on the weights of the bodies and velocity of the impact, and is determined by the rules for the colliſion of nonelaſtic bodies. Moreover, the reſtoration of the changed figures, how ſmall or great ſoever may have been the change, muſt cauſe an addition of velocity in the ball ſtruck equal to that received from the impact.

It follows alſo, that the effects of the ſtroke will be the ſame, whether both bodies be perfectly elaſtic, or one perfectly elaſtic, and the other perfectly hard, every thing elſe being given: for the figure of the elaſtic body muſt change till the bodies have obtained a common velocity, which depends on the weights and velocities before the ſtroke only, and will be the ſame as if the bodies were nonelaſtic; the reſtoration of the figure will in this, as well as in the former caſe, cauſe an increaſe of velocity in the ball ſtruck equal to that before communicated.

Although no ſubſtance in nature poſſeſſes perfect elaſticity, or is entirely deſtitute of it, yet there are ſeveral elaſtic and nonelaſtic bodies ſubject to experimental trials, wherein the laws relating to colliſion, are found to agree with matter of act, to a conſiderable degree of exactneſs.

IX.

The accelerating forces which communicate velocities to bodies, are as the moving forces directly, and the quantities of matter moved inverſely.

Since the accelerating force* is as the velocity generated in a given time, and by the laſt article the moving force is as the quantity of matter and velocity generated in a given time, it follows that the moving force is as the accelerating force, and the quantity of matter moved jointly; that is, the accelerating force is as the moving force directly, and the quantity of matter moved inverſely.

To illuſtrate this, let Q and q denote any two maſſes of matter: let m be the force or weight by which gravity impels q towards the earth's centre, and let M be any other

* Art. VI.

force which urges the maſs Q: then by the propoſition the ratio of the velocities generated in Q and q in a given time, or which is the ſame, the ratio of the forces which accelerate Q and q will be that of $M \times q : m \times Q$, wherefore if F be put for the force which accelerates Q, and f for the accelerating force of gravity, we ſhall have $F :$ $:: M \times q : m \times Q$, or $\frac{F}{f} = \frac{M}{m} \times \frac{q}{Q}$. Now it is manifeſt, that the values of the quantities $\frac{F}{f}$, $\frac{M}{m}$ and $\frac{Q}{q}$, depend only on the ratios of the numerators to the denominators; for which reaſon, the ſtandard or conſtant quantities f, m and q, with which F, M and Q are compared, may be *aſſumed $= 1$: this will give $F = \frac{M}{Q}$. Thus, let a maſs of matter = to that contained in 4 ounces of any ſubſtance, be impelled by a force = to the weight of 3 ounces, then the force which accelerates the maſs of 4 ounces will be $\frac{3}{4}$, when the acceleration of gravity is 1; or, in other words, it will generate in a given time three parts in four of the velocity which gravity generates during any given time in bodies which deſcend toward the earth's ſurface.

* Vid. Sect. II. Prop. VII.

When any quantity is ſaid to be given, it is meant, that the relation of it to ſome fixed quantity of the ſame ſort, conſidered as a ſtandard, is known; in like manner, when any quantity is ſought, it is required to find the relation of this unknown quantity to ſome fixed ſtandard of the ſame kind. All forces of acceleration are referred to gravity, that is, to the force whereby the earth accelerates adjacent bodies towards its centre. Time is referred to the earth's periodic revolution round its axis, whether it be conſidered as divided into hours or minutes, &c. Theſe two are definite ſtandards, common to all ages and climes, and it appears from the laws of motion, that ſtandards of ſpace and velocity alſo are determinable: but many difficulties in practice have hitherto prevented any methods from being carried into execution to fix theſe; ſo that different nations continue to make uſe of different meaſures to which they refer all other ſpaces: and even in the ſame nation theſe meaſures from the imperfection of the materials compoſing them, as well as from various accidents, have been found in a ſeries of ages liable to alteration.

The next ſection contains a few properties of ratios, which immediately relate to the mechanical propoſitions enſuing, as well as to the nature of the ſtandard quantities juſt deſcribed.

SECT. II.

CONTAINING SOME PROPERTIES OF RATIOS.

I.

TWO mathematical quantities of the ſame kind conſtitute a ratio.

Thus two lines, two ſurfaces, &c. conſtitute a ratio which may always be expreſſed by numbers, either commenſurable or incommenſurable with unity.

Phyſical quantities, which are in themſelves not mathematical, are nevertheleſs capable of mathematical relation, if they are meaſurable by ſpace or number; thus angles, velocities, times, and forces, and ratios themſelves, become capable of mathematical relation.

II.

The ratio of two mathematical or phyſical quantities may be compared with the ratio of two other quantities, though of a different kind from the former.

Thus when two bodies move uniformly for the ſame time, the ſpaces deſcribed vary with the velocities, and though ſpace and velocity, being different kinds of quantity, are not comparable, yet the ratio of the ſpaces is comparable with that of the velocities, theſe ratios being equal.

III.

If any quantity be divided by a quantity of the ſame ſort, the quotient becomes abſtract number.

Thus if U and v repreſent two velocities, then will $\frac{U}{v}$ be a number which is to 1 as $U : v$.

IV.

The ratio of any mathematical or phyſical quantities may be expreſſed by two numbers, if both terms of the ratio be divided by the conſequent, or by the antecedent.

Thus if F and f repreſent two forces, and both be divided by f, the ratio will become $\frac{F}{f} : 1$, which by the laſt propoſition is that of number to number.

V.

Any ratio may be repreſented by a fraction, the numerator of which is the antecedent, and denominator the conſequent of the ratio.

Hence, the addition and ſubſtraction of ratios may be performed by the multiplication and diviſion of fractions. Thus, when two bodies move uniformly, the ratio of the ſpaces deſcribed is equal to the ratio of the velocities added to the ratio of the times of motion, and is expreſſed thus at length, $S : s :: T \times V : t \times v$; or thus, $\frac{S}{s} =$

$\frac{T}{t} \times \frac{V}{v}$; which equation implies, that an abſtract number $\frac{S}{s}$, is equal to the product of two other abſtract numbers $\frac{T}{t}$ and $\frac{V}{v}$: and if the conſequents s, t, and v, be made unity, which aſſumption will not alter the values of the ratios, provided the antecedents S, T, and V vary proportionally, the arithmetic of theſe ratios will be hereby facilitated, for the laſt equation will become $\frac{S}{1} = \frac{T}{1} \times \frac{V}{1}$, or $S = T \times V$; S, T and V being abſtract numbers.

It muſt be obſerved in general, that whenever a mark of equality is interpoſed between heterogeneous quantities, as $S = T \times V$, no other equality is meant, but that which ſubſiſts between the ratios there expreſſed; that is, the ratio of the ſpaces, viz. $S : 1$, is equal to the ſum of ratios $T : 1$ and $V : 1$; ſo, ſince $T = \frac{S}{V}$, we have the ratio of the times, is equal to the ratio of the ſpaces diminiſhed by the ratio of the velocities. Moreover, when the mark of multiplication is interpoſed between heterogeneous quantities, it means the addition of two ratios, the antecedents of which are the terms expreſſed, and conſequents are unity.

VI.

Let $\frac{A}{a}$, $\frac{B}{b}$ and $\frac{C}{c}$, be three ratios, conſiſting of variable terms. If the relation of theſe quantities be ſuch, that when $\frac{C}{c}$ becomes unity, or the ratio of equality, $\frac{A}{a} = \frac{B}{b}$, alſo when $\frac{B}{b}$ becomes unity, if $\frac{A}{a} = \frac{C}{c}$, the propoſition aſſerts, that whatever be the magnitudes of $\frac{\mathrm{A}}{\mathrm{a}}$, $\frac{\mathrm{B}}{\mathrm{b}}$ and $\frac{\mathrm{C}}{\mathrm{c}}$, we always have $\frac{\mathrm{A}}{\mathrm{a}} = \frac{\mathrm{B}}{\mathrm{b}} \times \frac{\mathrm{C}}{\mathrm{c}}$.

For if not, let $\frac{A}{a} = \frac{B^{1+d}}{b^{1+d}} \times \frac{C^{1+e}}{c^{1+e}}$, then when $\frac{B}{b} = 1$, we have $\frac{A}{a} = \frac{C^{1+e}}{c^{1+e}}$, but by the hypotheſis, when $\frac{B}{b} = 1$, $\frac{A}{a} = \frac{C}{c}$: wherefore $\frac{C^{1+e}}{c^{1+e}} = \frac{C}{c}$ and $1 + e = 1$. This contradictory concluſion ariſes from denying the propoſition aſſerted, which is therefore true.

In general let $\frac{A^p}{a^p}$, $\frac{B^m}{b^m}$ and $\frac{C^n}{c^n}$ be three ratios, conſiſting of the variable terms $A\,a$, $B\,b$, and $C\,c$: if the relation of theſe quantities be ſuch, that when $\frac{C^n}{c^n} = 1$, $\frac{A^p}{a^p} = \frac{B^m}{b^m}$; alſo when $\frac{B^m}{b^m} = 1$, $\frac{A^p}{a^p} = \frac{C^n}{c^n}$; then whatever be the magnitudes of $\frac{A^p}{a^p}$, $\frac{B^m}{b^m}$ and $\frac{C^n}{c^n}$, we have $\frac{A^p}{a^p} = \frac{B^m}{b^m} \times \frac{C^n}{c^n}$: for if not, let $\frac{A^p}{a^p} = \frac{B^{m+e}}{b^{m+e}} \times \frac{C^{n+d}}{c^{n+d}}$, then when $\frac{B^m}{b^m} = 1$, we have $\frac{A^p}{a^p} = \frac{C^{n+d}}{c^{n+d}}$; but by the hypotheſis, when $\frac{B^m}{b^m} = 1$, $\frac{A^p}{a^p} = \frac{C^n}{c^n}$; wherefore $\frac{C^{n+d}}{c^{n+d}} = \frac{C^n}{c^n}$, and $n = n + d$, the leſſer equal to the greater, if d be of finite magnitude; and ſhould d be evaneſcent, we have $\frac{A^p}{a^p} = \frac{B^{m+e}}{b^{m+e}} \times \frac{C^n}{c^n}$, from which by proceeding in the ſame manner as before, it wou'd be deduced that $m + e = m$; that is, if e be of any finite magnitude, the leſſer is equal to the greater, which is impoſſible: it follows therefore, the quantities d and e being neceſſarily evaneſcent, that $\frac{A^p}{a^p} = \frac{B^m}{b^m} \times \frac{C^n}{c^n}$.

Cor. 1. The weights of bodies depend upon their magnitudes and denſities: and if W, w repreſent the weights of two bodies, M, m their magnitudes, and D, d their reſpective denſities, then when $M = m$, or $\frac{M}{m} = 1$, the weights will be as the denſities, or $\frac{W}{w} = \frac{D}{d}$; alſo, if $D = d$, the ratio of the weights $\frac{W}{w} = \frac{M}{m}$ the ratio of the magnitudes: wherefore by the propoſition, whatever be the quantities $\frac{W}{w}$, $\frac{M}{m}$ and $\frac{D}{d}$, we always have $\frac{W}{w} = \frac{M}{m} \times \frac{D}{d}$, or the ratio of the weights = the ſum of the ratios of the magnitudes and denſities.

Cor. 2. The ratio of equality is in quantity = 0, becauſe, being added to or ſubſtracted from any ratio, it cauſes neither increaſe nor diminution of the ratio. Thus, when $\frac{D}{d} = 1$, or when D and d are in the ratio of equality, we have $\frac{W}{w} = \frac{M}{m} \times \frac{1}{1} = \frac{M}{m}$.

Cor. 3. An inverſe ratio is that wherein the conſequent and antecedent are inverted, or if expreſſed by a fraction, it is when the numerator becomes the denominator, and the denominator the numerator, thus $\frac{m}{M}$ is the inverted ratio of $\frac{M}{m}$.

Cor. 4. An inverted ratio is equal in quantity to the direct ratio, but is negative in reſpect of it; for by adding the direct and inverſe ratios of two quantities together, the ſum becomes = 0, or the ratio of equality: thus the direct ratio $\frac{M}{m}$ added to the inverted ratio $\frac{m}{M}$ the ſum becomes $= \frac{mM}{Mm} = \frac{1}{1}$; thus ſince $\frac{W}{w} = \frac{D}{d} \times \frac{M}{m}$ we have $\frac{M}{m} = \frac{W}{w} \times \frac{d}{D}$, or the ratio of the magnitudes = the ratio of the weights added to the inverted ratio of the denſities; or ſince the inverted ratio of the denſities = — the direct ratio of the denſities, it follows, that the ratio of the magnitudes will be equal the difference between the ratios of the weights and denſities.

Cor. 5. When m, w and d, are each = 1, then will M,

D and W be abſtract numbers, and we have $M = \frac{W}{D}$ expreſſing the ratios deſcribed in the laſt Cor.

VII.

In the compariſon of the ratios which obtain between mathematical quantities of any ſort, the ſtandard to which each of thoſe quantities is referred, may be aſſumed = 1.

This will be beſt illuſtrated by an example. Let the weight, magnitude and denſity of any ſubſtance be reſpectively W, M and D. Theſe relative terms imply the conſideration of ſome ſtandard ſubſtance with the weight, magnitude and denſity of which W, M and D are compared. If the weight, magnitude and denſity of the ſtandard ſubſtance be w, m and d, we have in general $\frac{W}{w} = \frac{M}{m} \times \frac{D}{d}$.

Let a cubic inch of water be aſſumed as a ſtandard ſubſtance, then may w, m and d be each taken = 1, the correſponding terms being expreſſed proportionally to 1, and the equation expreſſing the relation of weight, magnitude and denſity of the ſubſtance, to thoſe of water, will be $\frac{W}{1} = \frac{M}{1} \times \frac{D}{1}$, or $W = M \times D$; W, M and D being abſtract numbers. This aſſumption will be moſt uſeful in general ſolutions, but in the application of them to particular caſes, ſince the weight of a cubic inch of water, which was aſſumed = 1, correſponds with none of our common weights, it is more convenient to aſcertain by experiment, what is the weight of a cubic inch of water, when referred to any of the ſtandard weights in uſe among us, viz. to ounces or grains, &c. and it appears upon trial, that the weight ſought is .57869 parts of an avoirdupoiſe ounce: this being therefore ſubſtituted in the preceding equation for w or 1, we have $\frac{W}{.57869} = M \times D$, or $W = .57869\ M \times D$, which expreſſes the relation of the weight, magnitude and denſity of the given ſubſtance, compared with the weight, magnitude and denſity of water, and accommodated to our ſtandard meaſures of weight and ſpace, that is, the weights being eſtimated in avoirdupoiſe ounces, and the magnitudes in cubic inches.

If inſtead of a cubic inch, m be made $=$ to a cubic foot, this will afford a greater facility in practice: in this caſe, $m = 1$ cubic foot, and water being ſtill conſidered as the ſtandard ſubſtance, $w =$ the weight of a cubic foot of water $= 1000$ ounces avoirdūpoiſe by experiment, if the ſpecific gravity of water be aſſumed $= 1000$, we have $\frac{W}{1000} = \frac{M}{1} \times \frac{D}{1000}$, or $W = M \times D$, which is the rule delivered in COTES's Hydroſtatics, Lect. 6.

The abbreviations made uſe of in the following propoſitions are theſe; F and f repreſent any two conſtant forces, V denotes the velocity generated by the force F in the time T, during which time the ſpace S is deſcribed by conſtant acceleration; v is the velocity generated during the time t, by the action of an accelerating force f, the ſpace deſcribed during the time t being s.

F and f are always underſtood to be accelerating forces proportional to the velocities generated in a given time.

SECT. III.

CONCERNING THE RECTILINEAR MOTION OF BODIES IMPELLED BY FORCES WHICH ACT UNIFORMLY.

I.

THE velocities generated in bodies by the action of conſtant forces, are as thoſe forces and the times in which they act jointly, or $\frac{V}{v} = \frac{F}{f} \times \frac{T}{t}$.

For when the times are the ſame, the velocities generated are as the forces of acceleration,* that is, when $\frac{T}{t} = 1$, $\frac{V}{v} = \frac{F}{f}$: and if the forces are the ſame, the velocities generated are as the times wherein the forces act; becauſe when the force is given, equal velocities † are generated in equal times, and conſequently the whole velocities acquired are as the times wherein the given force acts; that is, when $\frac{F}{f} = 1$, $\frac{V}{v} = \frac{T}{t}$; wherefore, both times and accelerating forces being different,‡ the velocities generated will be as the forces and the times of their action jointly: whatever therefore be the magnitude of $\frac{T}{t}$, $\frac{V}{v}$ and $\frac{F}{f}$, it is proved, that $\frac{V}{v} = \frac{F}{f} \times \frac{T}{t}$.

* 2d. Law of motion.

† Sect. I. Prop. IV.

‡ Sect. II. Prop. VI.

In this propoſition, F repreſents any conſtant force, which in the time T, generates a velocity V; f repreſents any other conſtant force, which in the time t, generates a velocity v; this propoſition, therefore, is wholly independent of the abſolute magnitudes of the forces, velocities and times, and conſequently, no abſolute quantity can be inferred from it. The uſe of the propoſition is, however, to compare any undetermined velocity, the conſtant force which produces it, and the time wherein it is acquired, with a known velocity, the known force by which it is generated, and the definite or ſtandard time wherin the known force acts reſpectively: the effects of this known force, as to the velocity produced and time of producing it, being determined by actual obſervation. In this caſe, therefore, f will repreſent a ſtandard force, v the velocity generated by it in the time t, all which might be † aſſumed equal 1, ſo as to make $V = F \times T$: and in general ſolutions unapplied to any abſolute known force, ſuch aſſumption is allowable, becauſe there is as much reaſon to denominate any one ſpace (for example $32\frac{2}{12}$ feet) unity, as any other.

† Sect. II. Prop. VII.

But ſince moſt phyſical propoſitions are either immediately or ultimately applied to the operation of natural forces, the effects of which are eſtimated by meaſures of ſpace, which, however arbitrarily fixed at firſt, ſtill continue in common uſe, it may be proper to deſcribe ſuch phyſical ſtandards of force, time and velocity, with which all other forces, times and velocities may be conveniently compared.

Of theſe three ſtandard quantities, two muſt be firſt given or fixed on at pleaſure, and the third obtained by actual obſervation. Now gravity is a force, the general and permanent nature of which renders it the moſt proper to be aſſumed as a ſtandard, to which all other forces may be referred: wherefore, this muſt be one of the given quantities, and may be * denoted by 1, every other force, with which it is compared, being repreſented by a number which is to 1, as that force is to gravity: on the other hand, bodies may acquire different velocities, deſcending by gravity in different times. We may therefore chuſe whether to aſſume a certain velocity, and denominating it 1, obſerve in what time the action of gravity generates this velocity; or we may aſſume a time = 1, and obſerve the velocity generated during the action of gravity for that time. The latter method, however, from the nature of the caſe is evidently preferable to the other; which will appear from the following conſiderations. Time is ne-

* Sect. II. Prop. VII.

ceſſarily

ceſſarily referred to the revolution of the earth round its axis, whether it be conſidered as divided into minutes or ſeconds, &c. this periodic revolution being liable to no alteration or variation whatever, and conſequently the meaſures of its parts, conſiſting of the portions of a given angle, deſcribed uniformly will be invariable, and therefore common to all ages, and places upon earth the moſt diſtant from each other.

We obſerve, from hence, that ſtandards of force and time are pointed out by nature herſelf, and ſo plainly as to have found univerſal reception among mankind: whereas, although ſtandards or invariable meaſures of ſpace and velocity, are alſo equally fixed by nature, yet the determination of them requiring ſkill in the theory, as well as the practical parts of mechanics, has among other cauſes prevented ſuch meaſures from becoming any where adopted; the reaſon of which is, that men find it neceſſary to weigh and to meaſure, before they become philoſphers; and in more improved times, philoſophy is of too little conſequence to alter what has been eſtabliſhed by centuries of continual uſe.

It is, however, neceſſary in phyſical enquiries, to obtain a ſtandard velocity, by actually obſerving, or inferring from ſome other actual obſerſervations, what velocity is generated by the force of gravity during a given time, for example one ſecond: this velocity being expreſſed in the meaſures of ſpace, which are in common uſe.

The determination of this ſtandard velocity will render the propoſition above demonſtrated, applicable to the compariſon of all other velocities generated in any different times, and by any conſtant forces, with ſuch velocities, as are generated by the force of gravity; the times and forces themſelves being at the ſame time compared: For ſuppoſe it to have been obſerved, that in a body's deſcent by the force of gravity for one ſecond, a velocity was generated which would, if uniformly continued, carry the body through a ſpace $= v$ in one ſecond of time: then referring to the general * equation $\frac{V}{v} = \frac{T}{t} \times \frac{F}{f}$, we have ‡ $t = 1$ ſecond, $f =$ the force of gravity $= 1$, and conſequently F and T abſtract numbers, wherefore $V = v \times T \times F$.

It has been hitherto ſuppoſed, for the ſake of illuſtration, that v may be obtained by obſerving the velocity

* Sect. III. Prop. I.

‡ Sect. II. Prop. VII.

city acquired in a body's deſcent for one ſecond of time; but ſince the acceleration of gravity cauſes a continual change in the velocity of the deſcending body, ſo as to render an eſtimation of the velocity at any particular point of time difficult and hitherto unattempted, the value of v, or the velocity acquired by a body which has deſcended for one ſecond, may be more eaſily aſcertained by the help of the next propoſition.

In the mean time, this will be the proper place to remark concerning the theorem already demonſtrated, that it is applicable to the motion of bodies acted upon by variable forces alſo, provided ths times wherein they act be taken ſo ſmall, that the forces may be regarded as conſtant.

Thus, let f, t and v repreſent any ſtandard force, time and velocity; and let F, T and V be other quantities of the ſame kind, which are compared with the former reſpectively; then however the force F may vary, yet if an element of time repreſented by $\dot{T}$, be taken for the time of its action, it will have the properties of a conſtant force, as far as regards the particle of time above deſcribed. Let $\dot{V}$ repreſent the velocity generated by the force F in the time $\dot{T}$, wherefore we have by the theorem $\frac{\dot{V}}{v} = \frac{F}{f} \times \frac{\dot{T}}{t}$; and if . v, f and t be aſſumed each $= 1$,* the equation will be $\dot{V} = F \times \dot{T}$.

II.

If a quieſcent body be impelled by any conſtant force acting upon it for a given time, the ſpace deſcribed will be to the ſpace deſcribed in the ſame time, by the body moving uniformly with the laſt acquired velocity in the ratio of one to two.

For let the given time be divided into equal evaneſcent inſtants, the number of which is n; then the ‡ velocity generated being as the time, and continuing uniform during any one inſtant, we ſhall have the ſpace deſcribed in any one inſtant proportional to the number of inſtants comprehended in the time of motion: ſo that if during the firſt inſtant the ſpace deſcribed be s, in the next inſtant the ſpace deſcribed will be $2s$, in the third $3s$, and in the three firſt inſtants the ſpace deſcribed will be $s + 2s + 3s$; ſo, in the n firſt inſtants the ſpace deſcribed will be $s + 2s + 3s \ldots ns = \frac{\overline{n+1} \times ns}{2}$: and ſince the * velocity laſt acquired is as the time, the force being given, and the ſpace deſcribed by any uniform velocity, is as the time and velocity jointly, it follows that the ſpace deſcribed by the laſt acquired velocity continued uniform for the time of the accelerated motion, will be as the ſquare of that time; ſo that if s be the ſpace uniformly deſcribed in the firſt inſtant of motion, $n^2 s$ will be the ſpace deſcribed in n inſtants with the velocity laſt acquired: wherefore the ſpace deſcribed by acceleration from quieſcence, is to the ſpace deſcribed uniformly with the laſt acquired velocity in the ſame time, as $n^2 s + ns : 2n^2 s$, or as $n + 1 : 2n$; and ſince gravity acts not by ſucceſſive impulſes, but by unceaſing acceleration, the magnitude of each inſtant muſt be diminiſhed, and conſequently their number increaſed, ſine limite; the laſt proportion therefore of $n + 1$, $2n$, will become that of $1 : 2$.

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* Sect. II. Prop. VII.

‡ Sect. III. Prop. I.

From this propoſition, the velocity which a body acquires in its deſcent by the acceleration of gravity for one ſecond, is eaſily obtained: for ſince this velocity † is ſuch as if continued uniform would carry the body in one ſecond through a ſpace twice greater, than that from which the body has deſcended, it follows, that to find the velocity acquired during any given time, for example one ſecond, it is only neceſſary, that the ſpace which a body deſcribes in its deſcent from quieſcence in one ſecond ſhould be obſerved. It is found from actual obſervation, (but more exactly from other methods,) that the ſpace through which a body deſcends from reſt in one ſecond, is equal 193 Engliſh inches, or 16 feet and one inch; and in this deſcent it appears from the propoſition,‡ that ſuch a velocity is acquired as would carry the body uniformly over $2 \times 16\frac{1}{12}$, or $32\frac{2}{12}$ feet in a ſecond of time; wherefore, if $16\frac{1}{12}$ feet, or 193 inches, be put equal l, then will the ſtandard velocity which was before repreſented by $u = 2l$.

* Sect. III. Prop. I.

† Sect. III. Prop. II.

‡ Sect. III. Prop. II.

By the firſt propoſition ‡ it is found, that in general $\frac{V}{v}=\frac{F}{f}\times\frac{T}{t}$, aſſuming therefore *$f$ and t each = 1, and ſubſtituting $2l$ for v, we have $\frac{V}{2l}=F\times T$ and $V=2l \times FT$.

‡ Sect. III. Prop. I.

* Sect. II. Prop. VII.

This propoſition is applicable to the motion of bodies impelled by variable forces, if the times of motion be aſſumed evaneſcent. Since $V=2lFT$, by taking the contemporary variations, we have $\dot{V}=2lF\times\dot{T}$, becauſe during the leaſt variation of the time $\dot{T}$ the force F is conſtant.

III.

The ſpaces which bodies deſcribe from reſt by the action of conſtant forces, are in a compound ratio of the velocities laſt acquired, and times of motion, or $\frac{S}{s}=\frac{V}{v}\times\frac{T}{t}$.

For the ſpaces deſcribed by the laſt acquired velocities continued uniform, are as thoſe velocities and the times of motion jointly: and the † ſpaces deſcribed by the accelerating forces acting conſtantly for equal reſpective times, are half the former ſpaces by the laſt propoſition.

† Sect. III. Prop. II.

Since in general $\frac{S}{s}=\frac{T}{t}\times\frac{V}{v}$, making § t = one ſecond, $v=2l$, and conſequently || $s=l$, we have $\frac{S}{l}=\frac{T}{1}\times\frac{V}{2l}$, or $S=\frac{V\times T}{2}$, which is an equation expreſſing the relation of S the ſpace deſcribed, and V, a meaſure of the velocity, being alſo a ſpace, which would be uniformly deſcribed by the laſt acquired velocity in the time 1: it appears that the ſpace deſcribed by the accelerated body, is equal to half of the ſpace V multiplied into an ab-

§ Sect. II. Prop. VII.

|| Sect. III. Prop. II.

ſtract number T, which expreſſes the number of ſeconds contained in the time of motion.

This propoſition is alſo true when applied to the motion of a body which is accelerated by a variable force, provided the time in which it acts be diminiſhed ſine limite, ſo that the force acting for an inſtant, may become conſtant.

Since $S = \frac{V \times T}{2}$, we have by taking the cotemporary variations $\dot{S} = \frac{\dot{V}T + V\dot{T}}{2}$; but $V = 2lFT$ and $\dot{V} = 2lF\dot{T}$, or $\dot{V}T = 2lFT\dot{T} = V\dot{T}$, which being ſubſtituted in the equation $\dot{S} = \frac{\dot{V}T + V\dot{T}}{2}$, it appears that $\dot{S} = V \times \dot{T}$ and $\dot{T} = \frac{\dot{S}}{V}$ in every ſcale, whatever be the value of the ſtandard ſpace l; $\dot{S}$ and V being referred to the ſame ſtandard ſpace, and $\dot{T}$ and V to the ſame ſtandard time: for every velocity reſpects two ſtandard meaſures, one of ſpace, and the other of time.

† Sect. III. Prop. II.

It appears therefore that the quantity $\dot{T}$, which is a number referred to any ſtandard 1, will be equal to the quotient of the ſpace deſcribed in the time $\dot{T}$, divided by the ſpace which would be uniformly deſcribed with the velocity of the body while it is deſcribing $\dot{S}$, in the time 1; whatever be the ſtandard ſpace l, to which S and V are referred.

IV.

Conſtant forces which accelerate bodies, cauſe them to deſcribe from reſt ſpaces which are as the forces and ſquares of the times wherein they act jointly, that is, $\frac{S}{s} = \frac{T^2}{t^2} \times \frac{F}{f}$.

For the ſpaces deſcribed, eſtimated from quieſcence, are as the velocities * laſt acquired and the times of motion jointly, or $\frac{S}{s} = \frac{T}{t} \times \frac{V}{v}$, and the velocities laſt acquired, are as the forces and the times of their acting jointly, † that is, $\frac{V}{v} = \frac{F}{f} \times \frac{T}{t}$; wherefore ſubſtituting $\frac{F}{f} \times \frac{T}{t}$ for its equal $\frac{V}{v}$, in the former equation, we have $\frac{S}{s} = \frac{T^2}{t^2} \times \frac{F}{f}$.

Cor, 1. If $F = f$, that is, if the forces are the ſame, it follows, that $\frac{S}{s} = \frac{T^2}{t^2}$, or the ſpaces are in a duplicate ratio of the times.

Cor. 2. Since ‡ $\frac{V}{v} = \frac{T}{t}$, when the force is given, ſubſtituting $\frac{V^2}{v^2}$ for $\frac{T^2}{t^2}$ in the laſt Cor. we have $\frac{S}{s} = \frac{V^2}{v^2}$, that is, the ſpaces are in a duplicate ratio of the velocities laſt acquired when the force of acceleration is given.

Cor. 3. Since $\frac{S}{s} = \frac{T^2}{t^2} \times \frac{F}{f}$, we have $\frac{F}{f} = \frac{S}{s} \times \frac{t^2}{T^2}$, that is, the ratio of the forces is compounded of the direct ratio of the ſpaces deſcribed, and the inverſe duplicate ratio of the times wherein the ſpaces are deſcribed from reſt, and conſequently when $T = t$, or the times being given, the ſpaces deſcribed are in the ſame proportion with the forces of acceleration.

Cor. 4. The propoſition contained in the laſt Cor. is applicable to the naſcent motion§, produced by a finite force any how variable, and juſt beginning to impel a body from quieſcence: thus a body will in a given time be attracted by the force of gravity through a greater ſpace near the earth's ſurface, than if the body were ſituated at the diſtance of the moon, in the proportion of 60 × 60 : 1, which is known from principles not immediately connected with the preſent ſubject; wherefore by the propoſition, the forces being as the ſpaces deſcribed when the time is given, it follows, that the force of gravity at the earth's ſurface, is to the force which it exerts at the diſtance of the moon, as 60 × 60 : 1.

Cor. 5. To apply this propoſition in order to eſtimate the force, time and ſpace in reference to the ſtandard mea-

* Sect. III. Prop. III.

† Sect. III. Prop. I.

‡ Sect. III. Prop. I.

§ Newt. Princip. Sect. I. Lemma X.

* Sect. II. Prop. VII.

ſures above deſcribed, ſince $\frac{S}{s} = \frac{T^2}{t^2} \times \frac{F}{f}$, let *$t = 1$ ſecond, and $f = 1$, the force of gravity, which acting for one ſecond, cauſes a body to deſcribe in its free deſcent $16\frac{1}{12}$ feet, or 193 inches, let this equal l, wherefore we have $\frac{S}{l} = T^2 \times F$, or $S = T^2 F l$.

V.

The conſtant forces which accelerate bodies from reſt are in a direct duplicate ratio of the velocities generated, and in an inverſe ratio of the ſpaces deſcribed, or

$$\frac{F}{f} = \frac{V^2}{v^2} \times \frac{s}{S}.$$

† Sect. III. Prop. I.
‡ Sect. III. Prop. III.

†For $\frac{F}{f} = \frac{V}{v} \times \frac{t}{T}$, and ‡ $\frac{t}{T} = \frac{V}{v} \times \frac{s}{S}$, ſubſtituting therefore $\frac{V}{v} \times \frac{s}{S}$, for its equal $\frac{t}{T}$ in the former equation, we have $\frac{F}{f} = \frac{V^2}{v^2} \times \frac{s}{S}$.

Cor. 1. The laſt acquired velocities are in a ſubduplicate ratio of the accelerating forces, and a ſubduplicate ratio of the ſpaces deſcribed jointly, for by the propoſition $\frac{V^2}{v^2} = \frac{F}{f} \times \frac{S}{s}$, or $\frac{V}{v} = \sqrt{\frac{F}{f}} \times \sqrt{\frac{S}{s}}$.

|| Newton. Princip. Sect. II. Prop. V.

Cor. 2. Let V and v repreſent the velocities of a body in different points of the ſame curve, or of two bodies moving in any two points of different curves, the centripetal forces being F and f, which retain the bodies revolving in their orbits. Let S and s repreſent the ſpaces, through which a body muſt move by the acceleration of the forces F and f continued conſtant, in order to acquire the velocities in the curves V and v reſpectively; || then will the centripetal forces be in a direct duplicate ratio of thoſe velocities, and an inverſe ratio of the ſpaces deſcribed: for according to the propoſition $\frac{F}{f} = \frac{V^2}{v^2} \times \frac{s}{S}$.

Cor. 3. If gravity be made the ſtandard * force $f = 1$, and ‖ $v = 2l$, then will $s = l$, and the equation will become $\frac{F}{1} = \frac{V^2}{4l^2} \times \frac{l}{S}$, or $V^2 = 4lFS$, and $V = \sqrt{4lFS}$.

* Sect. II. Prop. VII.

‖ Sect. III. Prop. I.

To exemplify: Suppoſe it were required to aſſign the velocity generated in a body, deſcending from reſt 3 feet along an inclined plane, the elevation of which above the horizon equal 30°: here $l = 16\frac{1}{12}$, $F = \frac{1}{2}$, $S = 3$, and the velocity required $= \sqrt{4 \times 16\frac{1}{12} \times \frac{1}{2} \times 3} = 9,82$ feet in a ſecond.

Cor. 4. This propoſition is alſo applicable to the action of forces however variable, provided the ſpaces, through which bodies are accelerated by them, be ſo diminiſhed, that the forces may be aſſumed conſtant.

Since by the propoſition $\frac{V^2}{v^2} = \frac{F}{f} \times \frac{S}{s}$, v, f, s and F being conſtant quantities, and V and S variable, taking the § leaſt cotemporary variations, $\frac{2V\dot{V}}{v^2} = \frac{F}{f} \times \frac{\dot{S}}{s}$, and making ‡ $f = 1$, $v = 2l$, and conſequently $s = l$, we have $2V\dot{V} = 4lF\dot{S}$.

§ Newt. Princip. Sect. VII. Prop. XXXIX.

‡ Sect. II. Prop. VII.

In order to accommodate this theorem to practice, let z be a ſpace through which a body muſt fall by the force of gravity, ſo that it may acquire the velocity V,† then will $V = \sqrt{4lz}$, and $V^2 = 4lz$, and $2V\dot{V} = 4l\dot{z}$.

† Cor. 3.

Subſtituting therefore $4l\dot{z}$, for $2V\dot{V}$ in the preceding equation, we have $4l\dot{z} = 4lF\dot{S}$, and $\dot{z} = F \times \dot{S}$, which equation expreſſes the relation of the homogenial quantities $\dot{z}$ and $\dot{S}$, F being a number which is to 1, as the accelerating force acting on the body while it is deſcribing $\dot{S}$ to the force of gravity.

Let AO be any ſpace which a body deſcribes by the action of a variable accelerating force, the quantity of which during the time wherein the body deſcribes the element of ſpace † $Oo = F$.

Fig. II.

† Sect. I. Prop. V.

Then ſuppoſing $AO = x$, $Oo = \dot{x}$, and z equal to a ſpace, which a body falls through by the acceleration of gravity, to acquire the velocity which the body deſcribing AO with the variable force poſſeſſes in the point O;

ſince $\dot{z} = F\dot{x}$, if F can be expreſſed in the terms of x, and the fluent of $F\dot{x}$ be obtainable, the velocity of the body, while it is deſcribing the ſpace Oo will be known, for any velocity will be determined, if we can aſcertain from what altitude a body muſt fall by the force of gravity to acquire that velocity. This altitude z is obtained from the equation $\dot{z} = F\dot{x}$, z being equal to the fluent of $F\dot{x}$: the velocity therefore with which the body deſcribes the ſpace $Oo = \sqrt{4l \times \text{fluent of } F\dot{x}}$, and if T repreſent the time of deſcribing AO, we have the time of deſcribing† Oo, that is, $\dot{T} = \frac{\dot{x}}{\sqrt{4l \times \text{fluent of } F\dot{x}}}$, and the time of deſcribing AO, or $T =$ the fluent of $\frac{\dot{x}}{\sqrt{4l \times \text{fluent of } F\dot{x}}}$.

This method of applying the Newtonian theory of acceleration to the ſolution of phyſical problems being the moſt commodious of any, eſpecially as it immediately reduces the ſolution to any required ſtandard of ſpace and velocity, ſhould be further deſcribed, as it will frequently hereafter be referred to. The quantity z which has been denominated the ſpace, through which a body muſt fall by the force of gravity to acquire the velocity in O, Euler,* who firſt brought this method into uſe, defines for the ſake of brevity, the ſpace due to the velocity at O, from the acceleration of the conſtant force 1, that of gravity for example. It being evident, that in the equation $\dot{z} = F\dot{x}$, it is intirely immaterial to what ſtandard meaſures z and x are referred, provided theſe meaſures be the ſame; if x, the ſpace deſcribed, be eſtimated in Engliſh feet, z will be alſo expreſſed in Engliſh feet, &c. It will therefore be unneceſſary in the ſolution of problems, to conſider the ſtandard quantity l, untill z the fluent of $F\dot{x}$ is aſcertained; this being effected, the ‡ velocity $V = \sqrt{4l \times \text{fluent of } F\dot{x}}$ will be expreſſed in whatever denomination l is taken.

This meaſure l is frequently eſtimated by foreigners in terms of the Rhynland foot, becauſe ſince the Rhynland foot is to the Engliſh, as 10000 : 9715, and $l = 16\frac{1}{12}$ Engliſh feet, it follows, that l expreſſed in Rhynland feet is equal 15.625,

† Sect. III. Prop. III.

* Motus Scient. Vol. 1. Sect. 3.

‡ Prop. V. Cor. 3.

15.625, or 15625 thouſandth parts of a Rhynland foot; therefore $4l = 62500$, and the velocity $V = 250 \sqrt{\text{fluent of } F\dot{x}}$ expreſſed by a thouſandth part of a Rhynland foot in a ſecond: this is here inſerted, becauſe ſome authors have adopted theſe numbers, without the neceſſary deſcription of the grounds upon which they are founded. The principal convenience of this method, is to avoid the extraction of the root $\sqrt{4l}$, which in no other meaſure, except that of the Rhynland foot, is a whole number. In the ſucceeding propoſitions, however, it will be more eligible to make uſe of the Engliſh feet and inches, to which we commonly refer the eſtimation of ſpace.

In the preceding propoſitions the quantity of matter moved has not been mentioned, becauſe the accelerating force, proportional ‖ to the velocity generated in a given time, being as the moving force ‡ directly, and the quantity of matter moved inverſely, the quantities of matter are therefore implied in the conſideration of accelerating forces. But ſome properties of motion require, that both the moving force and quantity of matter moved ſhould be ſeparately conſidered.

‖ Sect. I. Prop. VI.

‡ Sect. I. Prop. IX.

VI.

If bodies unequal in quantities of matter, be impelled from reſt through equal ſpaces, by the action of moving forces which are conſtant, theſe forces are in a duplicate ratio of the laſt acquired velocities, and the ratio of the quantities of matter jointly, or $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{Q}{q}$.

Let the moving forces be M and m, the quantities of matter moved Q and q, and the other notation remaining, we have * $\frac{V^2}{v^2} = \frac{F}{f} \times \frac{S}{s}$, and † $\frac{F}{f} = \frac{M}{m} \times \frac{q}{Q}$, or the accelerating forces are in a direct ratio of the moving forces, and an inverſe ratio of the quantities of matter moved;

* Sect. III. Prop. V.

† Sect. I. Prop. IX.

moved; wherefore ſubſtituting $\frac{M}{m} \times \frac{q}{Q}$ for its equal $\frac{F}{f}$, in the former equation, we have $\frac{V^2}{v^2} = \frac{M}{m} \times \frac{q}{Q} \times \frac{S}{s}$; or when $S = s$, that is, when the ſpaces deſcribed are equal $\frac{V^2}{v^2} = \frac{M}{m} \times \frac{q}{Q}$, and $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{Q}{q}$, that is, the ſpaces being in the ratio of equality, the moving forces will be in a ratio compounded of the duplicate ratio of the velocities, and of the quantities of matter moved.

Cor. When the ſpaces S, s are unequal, then $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{Q}{q} \times \frac{s}{S}$, that is, the moving forces are as the quantities of matter and ſquares of the velocities directly, and as the ſpaces deſcribed inverſely.

This, and all other propoſitions being deduced from the laws of motion, by regular and ſyſtematic reaſoning, ought not only to be ſtrictly conſiſtent among themſelves, but with matter of fact when examined by the ſevereſt trials; ſince any ſingle inſtance which could be produced of a diſagreement or inconſiſtency, would invalidate the whole theory of motion, by weakening the foundations on which it reſts: but it has not yet appeared, that there is any propoſition whatever, which is geometrically deduced from the axioms or received principles of motion, whether of the moſt complex or uncompounded nature, if it be reducible to accurate and deciſive examinations, but what is found coincident with the theory to any degree of preciſion, to the obſervation of which the human ſenſes are competent.

Many experiments, however, have been produced, as tending to diſprove the Newtonian meaſure of the quantities of motion communicated to bodies, and to eſtabliſh another meaſure inſtead of it, viz. the ſquare of the velocity and quantity of matter; and it immediately belongs to the preſent ſubject, to examine whether the concluſions which have been drawn from theſe experiments ariſe from any inconſiſtency between the Newtonian meaſures of force and matter of fact, or whether theſe concluſions are not ill founded, and ſhould be attributed to a partial examination of the ſubject: but ſome conſiderations concerning the principles of retarded motions ſhould premiſed.

In the experiments which have been made on the force of bodies, the loſs of motion from reſiſtance has been more attended to, than the communication of it by acceleration, and the reaſon probably aroſe from a want of adequate methods of ſubjecting accelerating forces, velocities acquired, and quantities of matter moved, to experimental trials; whereas the impact of bodies on ſubſtances, which they penetrate, by affording convenient opportunity for obſerving the depths to which bodies ſink before all motion is deſtroyed, regard being had to the velocities of impact, and the weight and form of the impinging body has ſeemed a more eligible (however imperfect) way of examining the principles of motion. This method is doubtleſs allowable, in order to eſtimate the force of moving bodies experimentally, if it be ſufficiently accurate; becauſe it is univerſally agreed, that (under certain reſtrictions not affecting the preſent queſtion) the laws of accelerated and retarded motions are mutually deducible from each other; and conſequently, what is found true in regard to the communication of motion to bodies accelerated, will be equally true when applied to the loſs of motion in bodies retarded, every thing elſe being the ſame, and vice verſa.

To exemplify: When a body deſcends by the force of gravity for three ſeconds, it acquires by conſtant acceleration a velocity of $3 \times 32\frac{2}{12} = 96\frac{1}{2}$ feet in a ſecond; alſo if a body be projected perpendicularly upwards, with a velocity of $96\frac{1}{2}$ feet in a ſecond, the whole velocity will be deſtroyed in three ſeconds; for the ſecond law of motion regards the change occaſioned in the velocities by retardation, as well as by acceleration, conſequently, if gravity acting in the direction of a body's motion, generates a velocity of $32\frac{2}{12}$ feet in a ſecond, the ſame force acting in a direction oppoſite to that wherein a body is moving, muſt deſtroy the ſame quantity of velocity in the ſame time, that is, $32\frac{2}{12}$ in each ſecond. In like manner, all the other properties which have been demonſtrated, concerning the motion of accelerated bodies, are ſhewn to belong to thoſe of retarded ones, the following circumſtances being attended to: If in any propoſition relating to accelerated motion, the force is conſtant, it follows, that when this is applied to retarded motion, the force of retardation muſt be conſtant: moreover, ſince in accelerated motions the ſpaces are eſtimated from quieſcence, ſo in retarded motions the bodies are ſuppoſed to move to quieſcence, that is, till all motion is deſtroyed by conſtant retardation.

In order therefore to apply the properties hitherto demonſtrated concerning accelerated motions to thoſe of retardation, the whole ſpaces deſcribed may be denominated S, s; and V, v, which before denoted the velocities laſt acquired, may now repreſent the initial velocities wherewith bodies are projected, T and t will repreſent the times of motion; moreover, F, f, will repreſent conſtant forces of retardation, meaſured by the velocities deſtroyed in a given time. If the force f, be that whereby gravity retards a body thrown perpendicularly upwards, it will become a ſtandard with which other forces of retardation may be compared; alſo M, which in accelerated motions denoted the moving force, now repreſents the reſiſting force of the ſubſtance, and is proportional to the quantity of motion deſtroyed in a given time: the retarding force will therefore be as the force of reſiſtance directly, and the quantity of matter in the moving body inverſely.

Theſe articles being premiſed, in order more fully to illuſtrate the ſubject, it is to be conſidered, that if a body projected with different initial velocities, be retarded by any given conſtant force, the whole ſpaces which the body deſcribes are in a duplicate ratio of the initial velocities; this follows, from what has already been demonſtrated *, and converſely, ſince when bodies are impelled by an accelerating force through various ſpaces, if theſe ſpaces are always as the ſquares of the laſt acquired velocities, it follows, that the force of acceleration is conſtant; ſo when a given body is projected with different velocities, and is retarded by a given force, if the whole ſpaces deſcribed be always in a duplicate ratio of the initial velocities, it is concluded that the force of retardation is conſtant.

* Sect. III. Prop. IV. Cor. 2. & p. 31.

It is from this argument inferred, that the force whereby blocks of wood, banks of earth, &c. reſiſt the penetration of bodies impinging on them, is conſtant: for it is obſerved, that the depths to which military projectiles of a given magnitude and weight, ſtriking againſt a body of this kind enter into its ſubſtance, are in a duplicate ratio of the initial velocities, which has been ſufficiently proved by Mr. Robins, who firſt aſcertained with certainty the velocities of military projectiles, and applied his method, among other uſeful purpoſes, to the diſcovery of the retardation, which bodies ſuffer by paſſing through reſiſting ſubſtances.

The forces of reſiſtance, which are oppoſed to the motion of bodies impinging on ſubſtances which they penetrate being granted conſtant, the propoſitions concerning ac-

acceleration already demonſtrated, may be applied to explain the motion of bodies, which having been projected with given initial velocities, are interrupted by ſuch obſtacles as blocks of wood, banks of earth, or others of a ſimilar kind.

For example: It has been demonſtrated, that bodies moving from reſt by the acceleration of conſtant forces, * deſcribe ſpaces which are as the accelerating ſorces and ſquares of the times jointly. By applying this propoſition to retarded motions, we ſhall have the whole ſpaces or depths, to which bodies impinging on the ſubſtances, penetrate, as the forces of retardation and ſquares of the times wherein the bodies move jointly.

* Sect. III. Prop. IV.

Moreover, it has been demonſtrated, that if different quantities of matter be impelled from reſt through equal ſpaces, [illegible] the † moving forces will be a ratio compounded of the duplicate ratio of the velocities laſt acquired, and the ratio of the quantities of matter moved. It is from hence inferred, that in retarded motions alſo, if different quantities of matter be projected againſt any of the ſubſtances above deſcribed, with different initial velocities, and the whole depths to which the bodies penetrate be equal, the forces whereby the ſubſtances reſiſt the progreſs of the impinging bodies, will be in a duplicate ratio of the initial velocities of impact, and the quantities of matter jointly: That is, if M, m be put to repreſent the forces of reſiſtance, Q. q the quantities of matter moved, V, v the initial velocities, the depths being equal, we have || $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{Q}{q}$, and if the depths be unequal, the equation will be $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{Q}{q} \times \frac{s}{S}$.

† Sect. III. Prop. VI.

|| Sect. III. Prop. VI.

By this propoſition we may examine ſome of the experiments concerning the force of moving bodies, and the concluſions deduced from them by Bernoulli, Leibnitz, Poleni, &c. againſt the meaſure of force delivered by Sir I. Newton, which he deſcribed in the following definitions.

‡ The quantity of motion is meaſured by the quantity of matter in a moving body and its velocity jointly.

‡ Newt. Princip. Def.

The moving forces whereby bodies tend toward centres of attraction, are as the quantities of motion generated in a given time.

It follows then from theſe definitions, that the moving forces acting for a given time, will be proportional to the quantities of matter moved, and velocities generated jointly; ſo that if the ratio of the moving forces be known, and we can find by experiment what velocities are generated in given bodies by the action of them for the ſame time, the quantities of motion generated in the bodies may be eſtimated according to Sir I. Newton's definition.

Moreover, ſince it is allowed, that the effects of a reſiſting force to deſtroy, are the ſame as thoſe of an equal moving force to generate motion in a given time, it follows, that if the ratio of two reſiſting forces be known, the quantities of matter in bodies which impinge on ſubſtances which they penetrate, and the velocities deſtroyed in a given time, will give the ratio of the quantities of motion deſtroyed according to Sir I. Newton's definition.

* Vid. infra notes on experiments.

In many of the experiments * alluded to, which have been greatly varied and multiplied, the reſiſting forces were made equal by cauſing ſpheres equal in magnitude to impinge on a given ſubſtance which they penetrated, and the ſpheres being of different denſities, it was obſerved in experiments, that whenever the denſities or weights of theſe equal ſpheres, were in an inverſe duplicate ratio of the initial velocities, the depths to which they penetrated would be equal.

The concluſions were theſe; the quantities of matter diſplaced by the moving bodies were equal, the depths to which the equal ſpheres penetrated being the ſame; moreover the whole motions which had been communicated to the bodies were deſtroyed: Theſe motions therefore wherewith the bodies impinged on the ſubſtances were equal; but by the experiments, the quantities of matter were in an inverſe duplicate ratio of the velocities, and conſequently the ſquare of the velocity into the quantity of matter, equal in both caſes; wherefore the quantities of motion deſtroyed, that is, the whole motion of the impinging bodies muſt have been as the ſquares of the velocities into the quantities of matter.

But it plainly appears, that the concluſion is not applicable to the Newtonian definition, according to which the moving force which generates motion in bodies, and it follows by what has preceded the reſiſting force by which the motion of bodies is deſtroyed, is proportional to the quantities of motion generated or deſtroyed in a given time reſpectively; and conſequently to eſtimate the quantities of motion deſtroyed, the time wherein the reſiſting forces act ſhould be equal. If therefore the times wherein the bodies in the experiment deſcribe the equal ſpaces, can be proved different, this will plainly ſhew that the quantities of motion deſtroyed cannot be inferred from the experiment, the different times of the bodies deſcribing the depths to which they ſink not being taken into the account: this will be eaſily proved, ſince from *propoſition 3d, we have univerſally the ſpaces deſcribed as the velocities laſt acquired and times jointly, this propoſition when applied to retarded motions will alſo be true; wherefore the ſpaces being given as in the experiment, the times will be inverſely as the initial velocities, which velocities being unequal from the conſtruction of the experiment, it follows that the times are unequal. This being the caſe, it is manifeſt that no concluſion can be drawn from theſe experiments concerning the quantities of motion deſtroyed, tending to prove any inconſiſtency between the Newtonian eſtimation of force and matter of fact. It is next to be ſhewn, that the experiments are ſtrictly conſiſtent with that meaſure, and with the theory in general.

From the propoſition above demonſtrated, we have in accelerated motions $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{Q}{q} \times \frac{s}{S}$, and conſequently $\frac{S}{s} = \frac{V^2}{v^2} \times \frac{Q}{q} \times \frac{m}{M}$; that is, the ſpaces deſcribed are in a duplicate ratio of the velocities laſt acquired and the quantities of matter moved, and an inverſe ratio of the moving forces; this propoſition being applied to retarded motions, it will be, the whole ſpaces or depths to which the impinging bodies ſink are in a duplicate ratio of the initial velocities, the ratio of the quantities of matter, and an inverſe ratio of the reſiſting forces. Wherefore, ſince in the experiment from the equality of the ſpheres' diameters $M = m$, we have $\frac{S}{s} = \frac{V^2}{v^2} \times \frac{Q}{q}$; but by the conſtruction of the experiment $\frac{V^2}{v^2} = \frac{q}{Q}$, ſubſtituting therefore $\frac{q}{Q}$ for $\frac{V^2}{v^2}$ in the former equation, it will become $\frac{S}{s} = \frac{Q}{q} \times \frac{q}{Q} = \frac{1}{1}$; wherefore $S = s$, or the depths to which the bodies penetrate, muſt be equal when ſpheres equal in diameter are projected againſt a given ſubſtance, the weights being in an inverſe duplicate ratio of the initial velocities, which we find entirely correſpondent with experiment.

In ſeems rational to ſuppoſe, independent of all theory, that in eſtimating the quantities of motion generated or

* Sect. III.

deſtroyed by given moving or reſiſting forces, regard muſt be had to the times wherein ſuch forces act; becauſe moving forces or thoſe of reſiſtance may be equal, and may generate or deſtroy quantities of motion varying in any aſſigned degree. For it is manifeſt, that a ſmall reſiſtance oppoſed to a moving body for a longer time, may deſtroy more motion than a greater force acting for a leſs time, which ſufficiently ſhews that the times wherein the moving and reſiſting forces act muſt either be equal, or muſt be taken into the account in eſtimating the quantities of motion generated or deſtroyed.

The next propoſition will demonſtrate, that when the times wherein motion is generated or deſtroyed, are equal, the moving and reſiſting forces, and their effects the quantities of motion generated or deſtroyed in a given time, will be as the quantities of matter moved, and the velocities acquired or deſtroyed jointly.

VII.

The moving forces which communicate, and the forces of reſiſtance which deſtroy the motion of bodies in the ſame time, will be in a compound ratio of the quantities of matter in the moving bodies, and velocities generated or deſtroyed, or $\frac{M}{m} = \frac{V}{v} \times \frac{Q}{q}$.

For * $\frac{F}{f} = \frac{V}{v} \times \frac{t}{T}$, that is, the accelerating or retarding forces, are as the velocities generated or deſtroyed directly, and the times wherein the bodies move inverſely; moreover, † $\frac{F}{f} = \frac{M}{m} \times \frac{q}{Q}$, that is, the accelerating or retarding forces, are as the moving or reſiſting forces directly, and the quantities of matter inverſely, wherefore we have $\frac{V}{v} \times \frac{t}{T} = \frac{M}{m} \times \frac{q}{Q}$, or $\frac{M}{m} = \frac{V}{v} \times \frac{Q}{q} \times \frac{t}{T}$, and when the times are the ſame, or $T = t$, it will be $\frac{M}{m} = \frac{V}{v} \times \frac{Q}{q}$, that is, the moving or reſiſting forces will be as the quantities of matter and velocities generated or deſtroyed in a given time: and the quantities of motion generated or deſtroyed will be in the ſame proportion.

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* Sect. III. Prop. I.

† Sect. I. Prop. IX.

This and the preceding propoſitions, when referred to experiments hereafter deſcribed, will appear ſtrictly coincident with them as they now do with each other, and with the theory in general.

The following truths have been derived from repeated experiments, and being deducible from the preceding theory, may be here inſerted as corollaries.

When muſket balls equal in weight and magnitude, impinge on a block of wood with different velocities, the reſiſting force being conſtant, we ſhall have the whole ſpaces through which the balls move in the wood until their motion is deſtroyed, as the ſquares of the velocities; * for $\frac{S}{s} \times \frac{M}{m} = \frac{V^2}{v^2} \times \frac{Q}{q}$, when $M = m$ and $Q = q$, that is, the ſubſtances on which equal bodies impinge being given $\frac{S}{s} = \frac{V^2}{v^2}$.

* Sect. III. Prop. VI.

Alſo, if balls of equal diameters but of different weights, impinge againſt a block with the ſame velocity, we have the depths to which they penetrate the block as the weights, for ſince $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{Q}{q} \times \frac{s}{S}$, when $\frac{M}{m} = \frac{V^2}{v^2} = 1$, it follows that $\frac{S}{s} = \frac{Q}{q}$.

If balls of the ſame kind of ſubſtance, that is, of the ſame denſity but of different diameters, impinge againſt a given block of wood, or the ſame bank of earth with equal velocities, the depths to which they penetrate will be directly as the diameters of the balls: for $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{Q}{q} \times \frac{s}{S}$, and when $V = v$, that is, when the velocities are the ſame $\frac{S}{s} = \frac{Q}{q} \times \frac{m}{M}$: or the ſpaces are in a direct ratio of the quantities of matter moved, and an inverſe ratio of the reſiſting forces, but $\frac{M}{m} = \frac{D^2}{d^2} \times \frac{R}{r}$, $\frac{R}{r}$ denoting the reſiſtances ariſing from the denſity and coheſion of the parts: moreover, $\frac{Q}{q} = \frac{D^3}{d^3}$ the ſpecific gravities being the ſame: by ſubſtituting therefore $\frac{D^2}{d^2} \times \frac{R}{r}$ for $\frac{M}{m}$, and $\frac{D^3}{d^3}$ for $\frac{Q}{q}$ in the equation $\frac{S}{s} = \frac{Q}{q} \times \frac{m}{M}$, we have $\frac{S}{s} = \frac{D^3}{d^3} \times \frac{d^2}{D^2} \times \frac{r}{R}$, that is, ſince in the ſame ſubſtance $R = r$, $\frac{S}{s} = \frac{D}{d}$, or the ratio of the ſpaces or depths to which the balls penetrate is the ſame as that of the diameters.

When the force of reſiſtance is not uniform, the ſame principle obtains in degree, *although according to various laws; for we obſerve that greater bodies always ſuffer leſs retardation than ſmaller ones of the ſame denſity, moving through the ſame reſiſting medium, and projected with a given initial velocity; becauſe though the force of reſiſtance increaſes with the increaſe of the body's magnitude, yet the weight increaſes in a higher proportion, the ſpecific gravity being the ſame, and therefore the retarding force which is meaſured by the force of reſiſtance directly, and the quantity of matter inverſely, is diminiſhed, as the magnitude of the moving body increaſes.

* Infra Sect. V.

It having been ſhewn, that the retarding and reſiſting forces, whereby the ſubſtances above deſcribed oppoſe the paſſage of bodies impinging on them is uniform, depending no ways on the velocities of motion; in order to apply theſe principles, the exact quantity of the retarding force exiſting in a given ſubſtance; or the proportion of it to ſome ſtandard force, ſuch as that of gravity, ſhould be aſcertained.

VIII.

If bodies projected with the ſame velocity, be retarded by different conſtant forces, theſe forces will be in an inverſe ratio of the whole ſpaces deſcribed by the projected bodies until all motion is deſtroyed, or $\frac{F}{f} = \frac{s}{S}$.

For * $\frac{V^2}{v^2} = \frac{S}{s} \times \frac{F}{f}$, and when $V = v$, as by the propoſition, it follows that $\frac{F}{f} = \frac{s}{S}$, that is, the forces of retardation are in an inverſe ratio of the ſpaces deſcribed.

For example: Let a body be projected on an inclined plane, in a direction contrary to that in which gravity accelerates bodies down the plane, and with a velocity of 144.467 inches in a ſecond; ſuppoſe the body thus projected aſcending along the plane to deſcribe 216 inches before its motion is deſtroyed, let it be required to aſcertain the retarding force which oppoſes its aſcent, that is, the proportion of it to the force of gravity. If the body were projected perpendicularly upward, with the given velocity of 144.467 inches in a ſecond, it would riſe to the altitude of $\frac{\overline{144.467}^2}{4 \times 193}$ † $= 27$ inches; and ſince it aſcends along the plane 216 inches, the retarding force on the plane will be to that of gravity, as 27 : 216, or 1 : 8, which is the proportion of the plane's height to the length alſo.

From this propoſition, having given the depth to which a body impinging againſt a ſubſtance with a given velocity penetrates the proportion of the retarding force to that of gravity may be determined.

For example: Mr. Robins found that a leaden ball of $\frac{3}{4}$ of an inch, or $\frac{1}{16}$ of a foot in diameter, impinging perpendicularly on a block of elm with a velocity of 1700 feet in a ſecond, penetrated into its ſubſtance to a depth of five inches, that is $\frac{5}{12}$ of a foot; wherefore ſince a body projected perpendicularly upwards, with a velocity of 1700 feet in a ſecond, would riſe to an altitude of $\frac{\overline{1700}^2}{4 \times 16\frac{1}{12}} =$ ‡ 44922 feet, we have the force whereby elm retards the ball to the force of gravity, as 44922 : $\frac{5}{12}$, or as 107813 to 1.

Moreover, || ſince $S = \frac{T \times V}{2}$, and therefore $T = \frac{2S}{V}$,

it

* Sect. III. Prop. V.

† Sect. III. Prop. V. Cor. 3.

‡ Sect. III. Prop. V. Cor. 3.

|| Sect. III. Prop. III.

it follows, in the preſent caſe, ſince $S = \frac{5}{12}$, and $V = 1700$, that the time of the balls deſcribing this ſpace of five inches $= \frac{2 \times 5}{12 \times 1700} = \frac{1}{2040}$ part of a ſecond.

In order to render this theory more general, it muſt be obſerved, that the reſiſtances oppoſed to ſpherical bodies which impinge on a block of wood, bank of earth, &c. depend not only on the tenacity or denſity of the parts, whereof the ſubſtances penetrated are compoſed, but upon the diameters of the impinging ſpheres; ſo that although the reſiſting and retarding forces be determined in any ſubſtance for a ſingle caſe, yet when the diameters and weights of the impinging bodies vary, the forces of reſiſtance and retardation oppoſed to the impact on the ſame ſubſtance will be different: by the preceding propoſition, however, we ſhall be enabled from a ſingle experiment, made on the retardation oppoſed by any given ſubſtance to a ſphere, whoſe weight and diameter is known, to infer the retardation in any other caſe, however the weights and diameters may vary. Suppoſe a given ball, impinging perpendicularly with the ſame velocity on any two ſubſtances, ſinks into one to the depth r, and into the other ſubſtance to the depth R, then will $\frac{R}{r}$‡ repreſent the ratio of the reſiſting forces whereby theſe given ſubſtances oppoſe the progreſs of the impinging body; theſe reſiſtances being independent of the magnitudes of the impinging bodies, and varying only with the denſity and coheſion of the parts whereof the ſubſtances conſiſt. $\frac{R}{r}$ may therefore be properly defined, the ratio of the abſolute reſiſting forces of the two ſubſtances.

‡ Sect. III. Prop. VIII.

Then if the diameters of the bodies impinging againſt the two ſubſtances be different, the whole reſiſtances being proportional to the quantities of motion deſtroyed in a given time, will be as the reſiſtances before determined $R : r$, and the ſquares of the diameters of the impinging ſpheres; that is, if D, d be the diameters of the ſpheres, the ratio of the whole reſiſtances $\frac{M}{m} = \frac{R}{r} \times \frac{D^2}{d^2}$, by help of which propoſition we may deduce a general theorem which will expreſs the relation of forces, which retard the progreſs of ſpheres impinging on any ſubſtances however the weights, diameters and ſpecific gravities may vary, provided the ratio $\frac{R}{r}$ be previouſly determined by experiment.

IX.

If ſpheres of different diameters and different ſpecific gravities, impinge perpendicularly on fixed obſtacles, the reſiſting forces of which are conſtant but of different quantities, the forces which retard the progreſs of the impinging ſpheres will be in a direct ratio of the abſolute forces of reſiſtance, and the joint inverſe ratio of the diameters and ſpecific gravities of the ſpheres.

For ſince in general* $\frac{M}{m} = \frac{F}{f} \times \frac{Q}{q}$, and by the preceding note $\frac{M}{m} = \frac{R}{r} \times \frac{D^2}{d^2}$, it follows that $\frac{F}{f} \times \frac{Q}{q} = \frac{R}{r} \times \frac{D^2}{d^2}$, and $\frac{F}{f} = \frac{R}{r} \times \frac{D^2}{d^2} \times \frac{q}{Q}$; but if N, n repreſent the ſpecific gravities of the ſpheres, we have $\frac{q}{Q} = \frac{d^3}{D^3} \times \frac{n}{N}$, and ſubſtituting $\frac{d^3}{D^3} \times \frac{n}{N}$, for its equal $\frac{q}{Q}$, in the equation, $\frac{F}{f} = \frac{R}{r} \times \frac{D^2}{d^2} \times \frac{q}{Q}$, it will become $\frac{F}{f} = \frac{R}{r} \times \frac{d}{D} \times \frac{n}{N}$; that is, the forces of retardation are in a direct ratio of the abſolute reſiſting forces, and an inverſe joint ratio of the ſpheres' diameters and ſpecific gravities.

As this propoſition only expreſſes the equality of ratios, no abſolute concluſion can be inferred from it relating to matter of fact, unleſs an experiment be firſt made on the retarding and reſiſting force of ſome ſubſtance, conſidered as a ſtandard, to which the forces of retardation and reſiſtance in other ſubſtances may be referred; as no ſtandard is in this caſe pointed out by nature, any one may be aſſumed; ſuppoſe it were the experiment of Mr. Robins before quoted, wherein the reſiſting ſubſtance is elm, the impinging body a ſphere of lead, the ſpecific gravity of which $= 11,35$, and diameter $\frac{3}{4}$ of an inch, or $\frac{1}{16}$ of a foot, and the force of retardation‡ $= 107813$; we have therefore $f = 107813$, $d = \frac{1}{16}$, $n = 11,35$, † and $r = 1$; then R being a number which is to 1, as the abſolute reſiſting force of the ſubſtance given is to that of elm, we have $\frac{F}{107813} = \frac{R}{r} \times \frac{n}{N} \times \frac{d}{D} = \frac{R}{1} \times \frac{11,35}{N} \times \frac{\frac{1}{16}}{D}$, or $F = \frac{107813 \times 11,35 \times \frac{1}{16} \times R}{D \times N} = \frac{76478R}{DN}$, the force which retards a ſphere impinging on a ſubſtance, the abſolute reſiſtance of which, in reference to that of elm, is as $R : 1$; the diameter of the ſphere being $= D$, and its ſpecific gravity N. It is next to be enquired, to what depth ſpheres impinging on reſiſting ſubſtances will penetrate, when the ſpheres' diameters, denſities, velocities of impact, and the abſolute reſiſtance of the ſubſtances whereon they impinge are given.

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* Sect. I. Prop. IX.

X.

The whole ſpaces or depths to which ſpheres impinging on different reſiſting ſubſtances penetrate, are in a ratio compounded of the duplicate ratio of the velocities of impact, the joint ratios of the diameters and ſpecific gravities of the ſpheres, and an inverſe ratio of the abſolute forces whereby the ſubſtances reſiſt the

‡ p. 39.

† Sect. II. Prop. VII.

the progreſs of the ſpheres, or $\frac{\mathrm{S}}{\mathrm{s}} = \frac{\mathrm{V}^2}{\mathrm{v}^2} \times \frac{\mathrm{D}}{\mathrm{d}} \times \frac{\mathrm{N}}{\mathrm{n}} \times \frac{\mathrm{r}}{\mathrm{R}}$.

†For in general $\frac{S}{s} = \frac{V^2}{v^2} \times \frac{f}{F}$, and by the laſt propoſition $\frac{f}{F} = \frac{r}{\mathrm{R}} \times \frac{N}{n} \times \frac{D}{d}$, ſubſtituting therefore theſe ratios for their equal $\frac{f}{F}$ in the equation $\frac{S}{s} = \frac{V^2}{v^2} \times \frac{f}{F}$, it will become $\frac{S}{s} = \frac{V^2}{v^2} \times \frac{D}{d} \times \frac{N}{n} \times \frac{r}{R}$.

† Sect. III. Prop. V.

From this propoſition and the experiment of Mr. Robins which has been aſſumed as a ſtandard, we may derive a general expreſſion for the value of S, or the depth to which an impinging ſphere will penetrate a given ſubſtance, provided the quantity $\frac{R}{r}$, which expreſſes the ratio of the abſolute reſiſtance of the ſubſtance to that of elm be firſt aſcertained from experiment.

For ſince in general $\frac{S}{s} = \frac{V^2}{v^2} \times \frac{D}{d} \times \frac{N}{n} \times \frac{r}{R}$, we have from Mr. Robins' ſtandard experiment $s = \frac{5}{12}$ part of a foot, $v^2 = \overline{1700}^2$, $d = \frac{1}{16}$ part of a foot, $n = 11.35$, and $r = 1$; wherefore $S = \frac{5}{12} \times \frac{V^2}{\overline{1700}^2} \times \frac{D}{\frac{1}{16}} \times \frac{N}{11.35} \times \frac{1}{R}$, that is, $S = \frac{V^2 D N}{4920223\, R}$ feet.

Suppoſe for example, an iron 24 pounder were fired againſt a bank of earth with an initial velocity of 1300 feet in a ſecond; and that the abſolute reſiſtance of the earth to that of elm, were as 1 to 11. Let it be required to aſſign to what depth the ball will penetrate in the bank of earth.

Since the ſpecific gravity of iron is $= 7.53$, and the balls weight $=$ 24 pound avoirdupoiſe, we have the diameter of the ball $= .46$ parts of an Engliſh foot, whereſore $D = .46$, $N = 7.53$, and $R = \frac{1}{11}$, then if the velocity of the cannon ball be that of 1300 feet in a ſecond, the depth to which it will penetrate into the earth $= \frac{V^2 \times D \times N}{4920223\,R} = \frac{\overline{1300}^2 \times .46 \times 7.53 \times 11}{4920223} = 13$ feet.

It appears, therefore, that an iron 24 pounder fired with its full charge, whereby it iſſues from the mouth of the cannon, with a velocity of 1300 feet in a ſecond, will penetrate into a bank of earth, upon which it impinges perpendicularly, to the depth of 13 feet, provided the reſiſtance of earth to that of elm be rightly aſſigned, and the bank be ſo near to the cannon that the air's reſiſtance ſhall not have ſenſibly decreaſed the initial velocity.

In eſtimating the proportion of the ſpaces deſcribed, times of deſcription, and velocities loſt by bodies, which having been projected with given initial velocities, are reſiſted by conſtant forces, the whole ſpaces have been hitherto neceſſarily ſuppoſed to be deſcribed: if it be required to aſſign the proportion of the ſpaces deſcribed by retarded bodies in any given times, eſtimating them from the firſt inſtant of projection, other rules will be neceſſary; theſe might be eaſily derived from geometrical conſtruction, but may be inveſtigated analytically in a more general manner.

XI.

Fig. III.

Let a body be projected from A in the direction A D, with the velocity which would be acquired by it in deſcending through the ſpace B C from reſt, by acceleration of gravity. Let the conſtant force F be oppoſed to the motion of the body, it is required to aſſign the time of deſcribing any ſpace A O, and the velocity of the body at the point O.

Let $BC = b$, $AO = x$, $V =$ the velocity in O, and let z be the ſpace through which a body muſt fall by the acceleration of gravity to acquire the velocity V. Then we § have $\dot{z} = -F\dot{x}$, and taking the fluents $z = -Fx$, which conſiſt of variable quantities only; theſe ſhould vaniſh together if they conſtituted the whole fluent, but ſince when $x = 0$, $z = b$, the conſtant quantity b muſt be added to $-Fx$, we have therefore $z = b - Fx$, and the velocity in the point O required $V = \sqrt{4lz} = \|\sqrt{4l \times \overline{b - Fx}}$. Moreover, if the time of deſcribing AO be put equal T, then will $\dot{T}$ denote the time of deſcribing the evaneſcent ſpace Oo,* therefore $\dot{T} = \frac{\dot{x}}{\sqrt{4l \times \overline{b - Fx}}}$, and the time $T = -\frac{\sqrt{b - Fx}}{F\sqrt{l}}$ being the variable part of the fluent, which requires a conſtant quantity $\frac{\sqrt{b}}{F\sqrt{l}}$ to be added to it, ſo that when $x = 0$, T may likewiſe equal 0. We have therefore the time ſought $T = \frac{\sqrt{b} - \sqrt{b - Fx}}{F\sqrt{l}}$ ſeconds.

Cor. 1. We obtain alſo the ſpace deſcribed in the time T, for by reducing the equation $x = 2T \times \sqrt{bl} - lFT^2$.

To exemplify this, ſuppoſe an elm plank of one inch in thickneſs being immoveably fixed, were perpendicularly ſtruck by a leaden ball of $\frac{3}{4}$ of an inch diameter, with a velocity of 1700 feet in a ſecond: the ball will paſs through the elm, and let it be required to aſſign what velocity will be deſtroyed during the ball's paſſage, or what will be the velocity of the ball immediately on its quitting the elm.

Here referring to the ſolution, we have $x = \frac{1}{12}$ of a foot, $l = 16\frac{1}{12}$, F† $= 107813$, and b ‡ $= \frac{\overline{1700}^2}{4 \times 16\frac{1}{12}} = 44922$ feet, being the altitude from which a body muſt deſcend freely by gravity to acquire the velocity of impact 1700 feet in a ſecond, and the velocity required $V = \sqrt{4l \times \overline{b - Fx}} = \sqrt{4 \times 16\frac{1}{12} \times \overline{44922 - \frac{107813}{12}}} =$

1520

§ Sect. III. Prop. V. Cor. 5.

|| Sect. III. Prop. V.

* Sect. III. Prop. III.

† Sect. III. Prop. VIII.

‡ Sect. III. Prop. V. Cor. 3.

1520 feet in a ſecond. From which it appears, that the velocity loſt by the impact is no more than 1700 — 1520 = 180 feet in a ſecond out of the original velocity of impact 1700.

It appears, moreover, that the time of the body's paſſing through the plank in the preceding caſe =

$$\frac{\sqrt{b} - \sqrt{b - Fx}}{\sqrt{l} \times F} = \frac{\sqrt{44922} - \sqrt{44922 - \frac{107813}{12}}}{\sqrt{16\frac{1}{12}} \times 107813} =$$

$\frac{1}{19328}$ part of a ſecond.

This will render it probable without further argument, that a muſket ball of certain dimenſions, may penetrate through an elm or deal plank ſet up vertically on its baſe, without other ſupport than its weight, and notwithſtanding the great velocity of the impact that it ſhall not ſenſibly move the plank from its vertical poſition. For even in the preceding caſe, but a ſmall portion of the ball's motion is communicated to the elm, and if any other ſubſtance of leſs thickneſs and reſiſting force be uſed in the experiment, the motion communicated to the plank ſet up vertically may be imagined ſo ſmall, as not to impel the line of direction out of the baſe. It might eaſily be computed what would be the exact effect of an impact of this ſort, (the neceſſary data being given) in raiſing the centre of gravity of the plank, and cauſing the whole to turn upon one of the edges which terminate the baſe; but the reaſon of the phenomenon above deſcribed is ſufficiently evident from the ſolution of the preceding problem.

Cor, 2. If the ſpace x, deſcribed in the time T, by a body which is retarded by a force F be given, the ſpace s, through which a body muſt fall by the force F to acquire the velocity of projection will be known. For let y equal the ſpace deſcribed by conſtant acceleration of the force F for T ſeconds, then will $y = \dagger lFT^2$, and $x = 2T \times \sqrt{bl} - lFT^2 = 2T\sqrt{bl} - y$, and $x + y = 2T \times \sqrt{bl}$; wherefore $\overline{x+y}^2 = 4\,T^2\,b\,l = \frac{4by}{F}$ becauſe $y = T^2 F l$, wherefore the ſpace through which a body muſt deſcend freely from reſt by the force of gravity to acquire the velocity of projection, that is the ſpace $b = \frac{\overline{x+y}^2}{4y}$

† Prop. IV. Cor. 5.

x

$\times F$, and the ſpace which muſt be deſcribed *by the acceleration of the force F to acquire the ſame velocity, that is the ſpace $s = \frac{\overline{x+y}^2}{4y}$.

* Sect. III. Prop. V.

If therefore a body be projected on an inclined plane in a direction contrary to that in which gravity accelerates bodies down the plane, and x be the ſpace deſcribed in the T firſt ſeconds, let y be the ſpace through which a body would deſcend from reſt on the plane during the ſame time T; then the body which was obſerved to deſcribe the ſpace x in T ſeconds, will in aſcending along the plane deſcribe the whole ſpace $= \frac{\overline{x+y}^2}{4y}$ before it loſes all motion.

For example: Suppoſe it to have been obſerved that a body projected along an inclined plane, in a direction contrary to that wherein gravity acts, deſcribed 868.5 inches during the two firſt ſeconds of its motion, the height of the plane being $\frac{1}{8}$ of its length. We have * the ſpace deſcribed on the plane by conſtant acceleration from reſt during two ſeconds, or $y = \frac{193 \times 4}{8} = 96.5$; and ſince by the queſtion $x = 868.5$, we have $x + y = 868.5 + 96.5 = 965$, and $\frac{\overline{x+y}^2}{4y} = \frac{\overline{965}^2}{4 \times 96.5} =$ 2412.5 inches, the whole ſpace along the plane to which the body will aſcend, if in the two firſt ſeconds it deſcribes 868.5 inches: and the time of deſcribing this whole ſpace $= \sqrt{\frac{2412.5 \times 8}{193}} = 10$ ſeconds.

† Sect. III. Prop. IV.

By this propoſition we may alſo aſcertain the ſpace deſcribed in the T laſt ſeconds, of the time wherein a body moves from reſt by a conſtant accelerating force F, provided the whole ſpace which the body deſcribes be given, and vice verſa, if the ſpace deſcribed in the laſt T ſeconds, and the force of acceleration be given, the whole ſpace may be aſcertained.

Thus ſuppoſe it to be obſerved that a drop of rain had during the laſt two ſeconds of its deſcent to the earth, deſcribed 400 feet. Let it be required to aſcertain the height from which it fell: referring to the ſolution we have $x = 400$ the ſpace obſerved, and the ſpace which would be deſcribed by the acceleration of gravity for two ſeconds $= 64\frac{1}{3} = y$ and $x + y = 464.333$, and ſince $F = 1$ we have the altitude required from which the rain deſcended $= \frac{\overline{x + y}^{2}}{4y} = 837.83$ feet.

It has been already obſerved, that all motion muſt be at firſt produced, and afterward increaſed or diminiſhed be continual acceleration or retardation. It being not conceivable that any natural body ſhould paſs from quieſcence into motion, without having poſſeſſed all the intermediate degrees of velocity. When any body therefore is ſtruck by another body, its parts give way in a greater or leſſer degree to the force of the impact.

Let the ſtriking ſubſtance and the body ſtruck be each of them ſpherical, and let the direction of the ſtroke be in a line which joins their centres.

Then, when the ſurfaces of theſe ſpheres are juſt in contact, that is, at the firſt inſtant of the impact, the diſtance of the centres will be half the ſum of their diameters; but as the figures of both ſpheres receive a change by the impact, the centres will thereby become nearer to each other than before, and during the time of their approach, the ſtriking ball by moving with a greater velocity than the ball ſtruck, will continue gradually to accelerate it till the centres are at their leaſt diſtance; at which inſtant the centres begin to move with a common velocity, the change of the ſpherical figures being then the greateſt.

This will be true whether the bodies be nonelaſtic or elaſtic; in the latter caſe, indeed, when the centres have become the neareſt poſſible to each other, the inſtant they are beginning to move with common velocity, the elaſtic force of the ſubſtances by endeavouring to reſtore the changed figure, cauſes the centres to recede from each other with a force equal to that which cauſed the change, provided the bodies be perfectly elaſtic.

Several conſiderations ariſe from hence: The time wherein this acceleration is generated, will depend on the force whereby the textures of the bodies reſiſt the force of the impact; the common velocity communicated, depends only on the weights of the bodies and the velocity of the impact: The forces of reſiſtance, varying with the diſtances of the centres, depend on the different textures of the ſubſtances; in moſt caſes the force muſt be evidently variable, but if the law of the variation can be aſcertained, the time wherein the velocity is communicated, as well as the approach of the centres, from the firſt inſtant of the impact, to the inſtant at which all acceleration ceaſes might be inveſtigated.

For example: When the reſiſting force is conſtant and of a determined quantity, having given the weights of the bodies and the velocity of the impact, from theſe data, the common velocity communicated and the time of communicating it, the change of the figure occaſioned by the impact, and the ſpace deſcribed by the body ſtruck before the two bodies have acquired a common velocity, may be aſcertained from the general principles of acceleration.

It is however to be obſerved, that in the following ſolution, the change in the figure of the body ſtruck being vaſtly greater than that of the ſtriking body, the latter is not conſidered.

XI.

HLGK is a block of wood, or other ſimilar ſubſtance, exiſting in a free ſpace without gravity. Let a body, for example a muſket ball, impinge perpendicularly on this block of wood when quieſcent in the direction IA, which line paſſes through the centre of gravity; having given the force wherewith the body reſiſts the progreſs of the ball, the quantities of matter contained in the ball and the block, and the velocity of the impact, it is required to aſcertain the velocity communicated. Fig. IV.

Let the velocity of the impact be ſuch as a body acquires in deſcending by gravity from reſt through the ſpace s, and let the quantity of matter in the block be

Q, that in the ball P, the reſiſting force of the block M; and ſuppoſe, that by the force of the impact the block has been propelled from its firſt poſition $GHLK$ to $CEDF$, during which time the ball has penetrated the block to the depth OB, having altogether moved over the ſpace $AO + OB$; then ſince the block being continually accelerated, and the ball continually retarded during the time wherein the ball ſinks into the ſubſtance of the block, the velocities of the two bodies will be different during that time: let v be the ſpace through which a body muſt fall by gravity to acquire the velocity of the ball at B, and v that ſpace which muſt be fallen through to acquire the velocity of the block at the ſame inſtant at which the ball has arrived at B. Moreover, let $AO = x$, $OB = y$, then will the ſpace $x + y$ be deſcribed by the ball while the block moves through the ſpace x, and the ball will deſcribe the elements of ſpace $\dot{x} + \dot{y}$ in the ſame time, wherein the block deſcribes the particle $\dot{x}$, and ſince the velocity of the ball at $O = \sqrt{\mathrm{v}}$, and the cotemporary velocity of the body ſtruck $= \sqrt{v}$, we have $\dot{x} + \dot{y} : \dot{x} :: \sqrt{\mathrm{v}}$ $\sqrt{v}$; becauſe the time of deſcribing the ſpaces $\dot{x} + \dot{y}$ and $\dot{x}$ is the ſame; wherefore $\dot{x} + \dot{y} = \sqrt{\frac{\mathrm{v}}{v}}\,\dot{x}$. We have therefore from the[*] principles of acceleration $\overline{\dot{x} + \dot{y}} \times \frac{M}{P}$[†] $= -\dot{\mathrm{v}}$, or ſince $\dot{x} + \dot{y} = \sqrt{\frac{\mathrm{v}}{v}}\,\dot{x}$, it follows that $\sqrt{\frac{\mathrm{v}}{v}}\,\dot{x}\,\frac{M}{P} = -\dot{\mathrm{v}}$: moreover ſince M is the reſiſting force whereby the body ſtruck deſtroys the ball's motion, $\frac{M}{Q}$ will be the force by which the block is [‡] accelerated, and it will follow [§] that $\frac{\dot{x}M}{Q} = \dot{v}$; ſubſtituting therefore $\frac{Q\dot{v}}{M}$ for $\dot{x}$ in the former equation, we have $\sqrt{\frac{\mathrm{v}}{v}} \times \frac{Q\dot{v}}{M} \times \frac{M}{P} = -\dot{\mathrm{v}}$, and by the reduction $\frac{\dot{v}}{\sqrt{v}} \times Q = -\frac{\dot{\mathrm{v}}}{\sqrt{\mathrm{v}}} \times P$, and taking the

[*] Sect. III. Prop. V. Cor. 4.

[†] Sect. I. Prop. IX.

[‡] Sect. I. Prop. IX.

[§] Sect. III. Prop. V. Cor. 4.

the fluents $\sqrt{v} \times Q = -\sqrt{\mathrm{v}} \times P$, which ſhould vaniſh together, but when $\sqrt{v}$, the velocity of the block $= 0$, $\sqrt{\mathrm{v}}$ the velocity of the ball $= \sqrt{s}$, the whole fluents therefore will be $\sqrt{v} \times Q = P\sqrt{s} - P\sqrt{\mathrm{v}}$, and $P\sqrt{\mathrm{v}} + Q\sqrt{v} = P\sqrt{s}$, that is, when both bodies move with a common velocity, or when $\sqrt{\mathrm{v}} = \sqrt{v}$, it will be $\overline{P+Q} \times \sqrt{v} = P\sqrt{s}$, and reſolving the equation into an analogy, $P + Q : P :: \sqrt{s} : \sqrt{v}$, or the ſum of the bodies is to the ſtriking body, as the velocity of the impact to the velocity communicated.

Cor. 1. Since during the whole acceleration, $P\sqrt{\mathrm{v}} + Q\sqrt{v} = P\sqrt{s}$, it follows, that the ſum of the moments is at every inſtant $= P\sqrt{s}$, the ſum of the moments before the ſtroke.

Cor. 2. As the quantity M, or the reſiſting force of the body ſtruck, enters not into the expreſſion for the common velocity, it follows, that whatever be the reſiſting force of the bodies, the weights of the bodies and the velocity of impact being the ſame, the velocity communicated will be given, being always equal to the velocity of impact, into the ſtriking body divided by the ſum of the bodies.

XIII.

Every thing remaining as in the laſt propoſition, let it be required to aſſign the depth O B, to which the ball penetrates into the ſubſtance of the block whereon it impinges.

Let the depth required $OB = y$, and $AO = x$, the other notation remaining as in the laſt propoſition, we * have $\overline{\dot{x}+\dot{y}} \times \frac{M}{P} = -\dot{\mathrm{v}}$, and taking the fluents $\overline{x+y} \times \frac{M}{P}$

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* Sect. III. Prop. XII.

$= -\mathrm{v}$; and ſince when $x + y = 0$, $\mathrm{v} = s$, it follows, that the whole fluents will be $\overline{x+y} \times \frac{M}{P} = s - \mathrm{v}$, or $x + y = \frac{sP - P\mathrm{v}}{M}$; and ſince $x = \frac{Qv}{M}$, we have $y = \frac{sP - P\mathrm{v} - Qv}{M}$, that is, when $\mathrm{v} = v$ or when the bodies have obtained a common velocity, $y = \frac{sP - \overline{P+Q} \times v}{M}$, ‡ but $\overline{P+Q}^2 : P^2 : s : v$, and $v = \frac{P^2 s}{\overline{P+Q}^2}$; wherefore $\overline{P+Q} \times v = \frac{\overline{P+Q} \times P^2 s}{\overline{P+Q}^2} = \frac{P^2 s}{P+Q}$, and $sP - \overline{P+Q} \times v = sP - \frac{P^2 s}{P+Q} = \frac{QsP}{P+Q}$, and the depth ſought $y = \frac{sP - \overline{P+Q} \times v}{M} = \frac{sPQ}{\overline{P+Q} \times M}$.

Cor. 1. Let s be the ſpace through which a body muſt deſcend by the acceleration of gravity, to acquire the velocity with which a body impinges on a block, the reſiſting force of which $= M$, and the weight $= Q$, let the weight of the impinging body $= P$, then if Q be infinite, that is, if the block be immoveably fixed, the depth to which the ball *penetrates $= \frac{sP}{M}$.

Cor. 2. If Q be of finite weight and P impinge on it, the depth to which it penetrates will be to the depth in that caſe wherein the block is immoveable, as the body ſtruck is to the ſum of the bodies, that is, $Q : P + Q$.

Cor. 3. To exemplify this, if a muſket ball of $\frac{3}{4}$ of an inch in diameter impinge againſt a block of elm, with a velocity of 1700 feet in a ſecond, and the block be nine times heavier than the ball, then $P = 1$, $Q = 9$ and $\frac{Q}{P+Q} = \frac{9}{10}$; moreover ſince the depth to which the ball would penetrate if the elm were † immoveable $= 5$ inches, it follows, that in the preſent caſe, when it is at liberty to move in free ſpace, the depth will be $\frac{5 \times Q}{P+Q} = \frac{5 \times 9}{10} = 4.5$ inches.

Cor.

‡ Sect. III Prop. XI. p. 51.

* Sect. III. Prop. VIII.

† Sect. III. p. 29. Robins' Gunnery, Vol. I. p. 152.

Cor. 4. Since the common velocity or that which is communicated to the block, is in this caſe $= \frac{1700 \times 1}{10}$ $= 170$ feet in a ſecond, it follows, that $v =$ the ſpace through which a body muſt fall by the acceleration of gravity to acquire the common velocity $= \frac{\overline{170^2}}{4 \times 16\frac{1}{12}} =$ *449.22 feet $= v$: moreover, ſince the retarding force $\frac{M}{P}$ oppoſed to † the ball $= 107813$, it follows, that the reſiſting force $M = 107813 \times P$, wherefore becauſe $x = \frac{Qv}{M}$ and $Q = 9P$, from ſubſtitution we have AO or $x = \frac{9Pv}{107813P}$ $= \frac{9 \times 440.22}{107813} = .0375$ part of a foot or .45 parts of an inch.

It appears therefore that the ſpace x or AO through which the elm block will be propelled before all acceleration ceaſes, that is, before the two bodies have acquired a common velocity, $= .45$ parts of an inch.

Cor. 5. The ſpace or depth to which the ball penetrates into the block $= \frac{sPQ}{\overline{P+Q} \times M}$, and the ſpace deſcribed by the block in the ſame time $= \frac{Qv}{M}$; this time will be aſcertained from the principles before demonſtrated, for in the motion of bodies accelerated from reſt, by a conſtant force, if S be the ſpace deſcribed, V the laſt acquired velocity, T the time wherein this velocity is generated, we have ‡ always $S = \frac{V \times T}{2}$ and $T = \frac{2S}{V}$; wherefore in the preſent caſe, the block being accelerated by the conſtant force $\frac{M}{Q}$, ſince the velocity acquired is 170×12 inches in a ſecond, and AO the ſpace deſcribed $=$ .45 parts of an inch, we have the time of deſcribing this ſpace $= \frac{2 \times .45}{170 \times 12} = \frac{1}{2267}$ part of a ſecond.

* Sect. III. Prop. V. Cor. 3.

† Sect. III. Prop. VIII. p. 39.

‡ Sect. III. Prop. III.

SECT. IV.

CONCERNING THE RECTILINEAR MOTION OF BODIES ACTED ON BY FORCES WHICH VARY IN SOME RATIO OF THE DISTANCES FROM A FIXED POINT.

THE laws obſerved in the motion of bodies which are uniformly accelerated or retarded, will ſufficiently explain all the phenomena which ariſe from the action of conſtant forces, or even of variable ones in the very beginning of the motion produced by them. Of this ſort is the force of gravity, which muſt in ſtrictneſs be accounted a variable force, the law of variation being that of the inverſe ſquares of the diſtances from the earth's centre. But any ſpaces through which bodies can actually deſcend, are ſo ſmall when compared with the radius of the earth, that the greateſt ratio of the ſquares of thoſe diſtances from the earth's centre differs little from that of equality. It is for this reaſon, that the force of gravity,

vity, which accelerates the deſcent of bodies and retards their aſcent, has been regarded according to Galilæo's hypotheſis as conſtant. This aſſumption, however, would be leſs allowable than it is, were we not enabled by the laws of variable forces to demonſtrate how ſmall a deviation from the truth is occaſioned by it.

Gallilæo's diſcoveries in mechanics related chiefly to the operation of conſtant forces, but were not ſufficient for the eſtimation of the effects produced by thoſe which are variable; for this purpoſe, methods are required, unknown till Sir I. Newton applied his invention of fluxions, to determine from the neceſſary data, the ſpaces deſcribed, times of deſcription, and velocities acquired by bodies acted on by forces varying according to any aſſigned law.

The properties of variable forces being deducible from thoſe of conſtant ones, acting in evaneſcent times, or impelling bodies through evaneſcent ſpaces, have been inſerted as corollaries to the propoſitions in the preceding ſection: from theſe principles, the effects of variable forces acting during times which are finite, and impelling bodies through finite ſpaces, are to be inferred.

The extent of the Newtonian theory of acceleration appears from conſidering that there are few motions produced either by the operations of nature or of mechanic art, but what are the effects of variable forces. Theſe may either accelerate or retard the motion of bodies, and may obſerve different laws in regard to the intenſity with which they act: for example, theſe forces may vary according to ſome power of

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The propoſition contained in Cor. 4. Sect. iii. which is applied to inveſtigate the effects of variable forces, may be inferred from propoſition the vth, otherwiſe as follows: it appears in the propoſition, that $\frac{V^2}{v^2} = \frac{F}{f} \times \frac{S}{s}$. Let s be the ſpace through which a body muſt move from reſt by the acceleration of the force f, ſo as to acquire a velocity v equal to V, which is generated by the force F acting on a body moving from reſt through the ſpace S: then $V = v$ and $F \times S = f \times s$: here, if f be the force of gravity, and s the ſpace through which a body muſt deſcend freely by that force from reſt, ſo as to acquire the velocity v or V, f may be denoted by 1, and F by a proportional number. This will give $FS = s$, and F being conſtant during an inſtant, $F\dot{S} = \dot{s}$, the ſame as in Cor. 4. where $\dot{z}$ repreſents the ſame quantity, as is here ſignified by $\dot{s}$. When S decreaſes while s increaſes, their increments will be negative in reſpect of each other; and the fundamental equation will become $\dot{s} = -F\dot{S}$, or $\dot{z} = -F\dot{x}$.

the diſtances from a fixed point, or according to ſome power of the velocities, beſides innumerable other laws of variation which obtain in particular caſes: The enſuing ſection contains properties of rectilinear motion, cauſed by the action of forces, which vary in ſome ratio of the diſtances from a fixed point, being ſuch as chiefly relate to natural phænomena.

I.

Let C repreſent a centre, toward which bodies are accelerated by forces which are always in a direct proportion of their diſtances from C. Suppoſing the force at any given point G to be $=$ F, it is required to aſcertain the velocity acquired by a body, which has moved from quieſcence at A through the ſpace AO. Fig. V.

Let any ſtandard force 1, the earth's gravity for example, be that to which F is referred, $CA = a$, $CG = r$, $AO = x$, alſo let V be the velocity required; then ſince by the problem, the force at O is to that at G as CO to CG, that is, as $a - x : r$, we have the force at $O =$ $\frac{F \times \overline{a - x}}{r}$.

Let z be the ſpace through which a body muſt move from reſt by the acceleration of the ſtandard force 1, continued conſtant ſo as to acquire the velocity at O; this will give *$\dot{z} = \frac{F \times \overline{a\dot{x} - x\dot{x}}}{r}$ and $z = \frac{F \times \overline{2ax - x^2}}{r}$, which, ſince when $x = 0$, $z = 0$, is the entire fluent:

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* Sect. III. Prop. V. Cor. 4.

wherefore the velocity $V = \sqrt{\frac{F \times \overline{2ax - xx}}{2r}}$, or $V = \sqrt{\frac{F}{2r}} \times \sqrt{2ax - xx}$: moreover, if T be made to repreſent the time of deſcribing AO, we have *$\dot{T} = \sqrt{\frac{2r}{F}} \times \frac{\dot{x}}{\sqrt{2ax - xx}}$, and $T = \sqrt{\frac{2r}{F}} \times$ fluent of $\frac{\dot{x}}{\sqrt{2ax - xx}}$, that is, $T = \sqrt{\frac{2r}{Fa^2}} \times$ into an arc of a circle the verſed ſine of which $= x$, and radius $= a$. When $x = a$, that is, when $AO = AC$, the fluent will become the time of deſcribing the whole ſpace AC, which time will $= \sqrt{\frac{2r}{Fa^2}}$ $\times$ into the quadrant of a circle the radius of which is $= a$. Let $p = 3.14159$, &c. the quadrant of the circle will be $\frac{pa}{2}$, and the time of deſcribing $AC = \sqrt{\frac{2r}{Fa^2}} \times \frac{pa}{2} = \frac{p \times \sqrt{r}}{\sqrt{2F}}$.

* Sect. III, Prop. III.

If the body begins to deſcend from any other point B, the ſolution continues the ſame in every ſtep, and the time of deſcribing the whole ſpace $BC = \frac{p \times \sqrt{r}}{\sqrt{2F}}$, which, if the diſtance r or CG and F the force at G be unaltered, will be equal to that wherein the body deſcribed the ſpace AC.

Cor. 1. If C be a given centre of force, varying as deſcribed in the problem, and the force at any given diſtance CG, continue conſtantly equal to F, then bodies deſcribing from reſt any ſpaces BC, GC, AC, DC, will all arrive at the centre in equal times, the time of the deſcription being $= \frac{p \times \sqrt{CG}}{\sqrt{2F}}$.

Cor. 2. If the given force F be referred to the earth's gravity, the latter may be denoted by 1, wherefore F will be

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a number which is to 1, as the force at G to the force of gravity: then referring the times of deſcent to 1 ſecond, and the ſpaces deſcribed to feet or inches, let $l = 16\frac{1}{12}$ feet, or 193 inches, and we ſhall have the time of deſcribing from reſt any ſpaces BC, GC, or *$AC = \frac{p \times \sqrt{CG}}{\sqrt{8Fl}}$. * Sect. III. Prop. V. p. 28.

Cor. 3. Let $ADGCE$ repreſent a cycloid, and let a body ſuſpended from the line or axis SC vibrate in it, and ſuppoſe it were required to aſſign the time of the pendulum's deſcribing any arc DC, beginning its motion from D, C being the loweſt point through which the axis SC paſſes: let AC or $SC = L$: In order to aſcertain the force which acts on the pendulum in any given point G, let the line GE be drawn perpendicular to HC, interſecting the generating circle in K: then to find the force at G, the body's weight being repreſented in quantity and direction by any conſtant line, which is perpendicular to the horizon, let the line CH (being $\frac{1}{2}$ the axis) denote the weight of the pendulum, this may be reſolved into two CK, HK, whereof CK being parallel to a line touching the cycloid in the point G, accelerates the motion of the pendulum in the curve. The other force HK, acting in a direction perpendicular to the former, tends only to ſtretch the ſtring and affects not the acceleration. The force, therefore, which accelerates the pendulum will be that part of its weight which is expreſſed by the fraction $\frac{CK}{CH}$; the force alſo which accelerates the pendulum in any other point of its vibration D, will be that part of gravity which is expreſſed by the fraction $\frac{CO}{CH}$, wherefore the force at G will be to the force at D, in the proportion of $\frac{CK}{CH}$ to $\frac{CO}{CH}$, or as CK to CO, that is, as CG to CD; becauſe the cycloidal arcs are double to the correſponding chords of the generating circle. The forces which act at G and D being in the direct proportion of the diſtances of thoſe points from the point C, the time of deſcribing either GC or DC will be obtained from the preceding propoſition, if the length of the pendulum SC be given, and the action of the force at any given diſtance from C be aſcertained. Suppoſe CG to be $\frac{1}{2}$ AC or SC, then will CK be $\frac{1}{2}$ HC, Fig. VI.

and the force at $G = \frac{1}{2}$ of the body's weight or $F = \frac{1}{2}$, and ſince the time of deſcribing any *ſpace $DC = \frac{p \times \sqrt{GC}}{\sqrt{8Fl}}$, by ſubſtituting $\frac{L}{2}$ for GC, and $\frac{1}{2}$ for F, we have the time of deſcribing the arc GC or $DC = \frac{p \times \sqrt{L}}{\sqrt{8l}}$, and the time of one entire vibration $= \frac{p \times \sqrt{L}}{\sqrt{2l}}$.

* Cor. 2.

Cor. 4. If the length of a pendulum, and the force of acceleration at the greateſt diſtance from the loweſt point A, continue unaltered and equal to F, the pendulum will perform its vibrations in the longeſt and ſhorteſt arcs in the ſame time, namely, in the time $\frac{p \times \sqrt{L}}{\sqrt{2Fl}}$. Pendulums of different lengths will vibrate in times which are in a direct ſubduplicate ratio of their lengths, and an inverſe ſubduplicate ratio of the forces which accelerate them at any diſtances from the loweſt point, which are in a given proportion to the whole lengths.

Cor. 5. Let F be the force which acts upon the pendulum when at the greateſt diſtance from the loweſt point: Since $T = \frac{p \times \sqrt{L}}{\sqrt{2Fl}}$, T being the time of one vibration, and ‡ ſince $\sqrt{\frac{L}{2Fl}}$ is the time of a body's deſcribing the ſpace $\frac{L}{2}$ from reſt, by the action of a conſtant force F, by reſolving the equation into an analogy we have $T : \sqrt{\frac{L}{2Fl}} :: p : 1$, that is, the time of one vibration is to the time of a body's deſcribing from reſt half the pendulum's length by the conſtant acceleration of the force F, as the circumference of a circle is to its diameter.

† Sect. III.
Prop. IV.
Cor. 5.

Cor. 6. If a pendulum L could vibrate in the chords of a circle, the time of one vibration would be equal to that wherein a body deſcending from reſt deſcribes $8L$, that is, the time of one vibration $= \sqrt{\frac{8L}{l}}$, F being $= 1$: a pendulum of the ſame length would vibrate in a cycloidal arc in the time $\frac{p \times \sqrt{L}}{\sqrt{2l}}$, the force F being 1; wherefore we have the following proportion: the time wherein a pendulum would vibrate in the chords of a circle is to the time of its vibration in the cycloidal arc, as $\sqrt{\frac{8L}{l}}$ to $\frac{p \times \sqrt{L}}{\sqrt{2l}}$, that is, as 4 : p, or as 4 diameters of a circle to the circumference.

Cor. 7. The general value for the time of one vibration being $\frac{p \times \sqrt{L}}{\sqrt{2Fl}}$, if in any given part of the earth, the force of gravity be denoted by 1, and the ſpace through which a body deſcends from reſt in one ſecond be equal l, we ſhall obtain the length of a pendulum which vibrates ſeconds in that latitude; for ſince $T = \frac{p\sqrt{L}}{\sqrt{2Fl}}$, we ſhall have, T and F being $= 1$, $L = \frac{2 \cdot l}{p^2}$. If $l = 193$ Engliſh inches in that latitude (for example at London) then $L = \frac{2 \times 193}{3{\cdot}1459 \times 3{\cdot}1459} = 39{,}2$ Engliſh inches. If the force of gravity at the equator be to that at London as $F : 1$, then the length of a pendulum which vibrates ſeconds at the equator muſt be $F \times 39.2$ Engliſh inches.

If therefore it be obſerved, what is the proportion of the lengths of two pendulums which vibrate ſeconds at any two latitudes, the proportion of theſe lengths will give the proportion of the forces of gravity at thoſe places, the effects of the earth's centrifugal force on the times of vibrations not being here conſidered.

Cor. 8. From the third corollary we may obtain a rule for correcting the pendulum of a clock: for the time of one vibration $T = \frac{p\sqrt{L}}{\sqrt{2l}}$, and $T^2 = \frac{p^2 L}{2l}$, of which quantities T and L only are variable: wherefore taking the leaſt cotemporary variations of their hyperbolic logarithms, we have $\frac{2\dot{T}}{T}=\frac{\dot{L}}{L}$, where $\dot{T}$ repreſents an error in time cauſed by the variation in the pendulum's length $\dot{L}$: Theſe variations are in theory leſs than any aſſignable quantities of the ſame ſort, but in practice the rule will be ſufficiently true when the errors bear a very ſmall proportion to the whole quantities. Suppoſe for inſtance, a pendulum vibrating ſeconds, was obſerved to loſe 10 ſeconds, in 9 hours or 32400 ſeconds. Let it be required to aſſign how much the pendulum muſt be ſhortened, ſo as to keep true time. Here $T=1$, $\dot{T}=\frac{10}{32400}$, and $L=39.2$, then from the rule we have the alteration ſought $\dot{L}=L\times\frac{2\dot{T}}{T}=\frac{39.2\times 20}{32400}=\frac{1}{41.3}$ part of an inch.

‡ Newt. Princip. Prop. LXXIII.

Cor. 9. ‡ Since a body ſituated within the earth's ſurface, is attracted by a force which varies in a direct ratio of the diſtance from the centre, the time wherein a body would deſcend freely from the ſurface to the earth's centre, will be $\frac{1}{2}$ the time in which a pendulum, equal in length to the earth's radius, would perform its leaſt vibrations in a circular arc and vibrations of any length in a cycloid; that is, making r equal to the earth's radius, the time would be $\frac{p}{2}\times\sqrt{\frac{r}{2l}}=$ ſeconds, or twenty one minutes thirteen ſeconds.

Cor. 10. The relation of the ſpaces deſcribed, times of deſcription, and velocities acquired, may be geometrically conſtructed thus.

Fig. VII.

Let AC repreſent the ſemicycloidal arc drawn out into a ſtrait line, and the force at A being equal to that of gravity, and C the loweſt point of the arc: then ſuppoſe the pendulum to begin its vibration from any point D, the time of deſcribing DC will be equal to that of deſcribing the whole arc AC. With the centre C and diſtance CD, deſcribe the quadrant DB, then ſuppoſing DO, DM to be two ſpaces deſcribed during the motion of the body from D to C, the † velocities in the points O and M will be as the ſines OF, MN: for the velocity* at $O=\sqrt{\frac{2CD\times DO-DO^2}{2CA}}$, and the velocity at $M=\sqrt{\frac{2CD\times DM-DM^2}{2CA}}$, wherefore the velocity

† Newt. Princip. Prop. XXXVIII.

* Vid. ſupra p. 58.

velocity at O will be to that at M, as $\sqrt{\frac{2CD \times DO - DO^2}{2CA}}$ to $\sqrt{\frac{2CD \times DM - DM^2}{2CA}}$, or multiplying both by $\sqrt{2CA}$, as $\sqrt{2CD \times DO - DO^2}$ to $\sqrt{2CD \times DM - DM^2}$, that is, by the properties of the circle, as OF to MN: Moreover, the time of deſcribing DO will be to the time of deſcribing DM, as the arc DF to the arc DN, for the time of deſcribing $DO = \sqrt{\frac{2CA}{CD^2}} \times$ into the arc of a circle, the radius of which $= CD$, and verſed ſine $= DO$, that is, the time $= \sqrt{\frac{2CA}{CD^2}} \times$ arc DF; for the ſame reaſon, the time of deſcribing $DM = \sqrt{\frac{2CA}{CD^2}} \times$ arc DN; wherefore the proportion of theſe times is as $\sqrt{\frac{2CA}{CD^2}} \times DF$ to $\sqrt{\frac{2CA}{CD^2}} \times DN$, or as DF to DN.

II.

Let C be a centre, toward which bodies are attracted with forces which are inverſely proportional to the ſquares of their diſtances from C. Moreover, let the force, which acts at any given point G, be F, referred to a ſtandard force 1, ſuch as that of the earth's gravity. If a body begins to deſcend from a point A, by the action of the centripetal force, it is required to aſcertain the velocity acquired at any given point O of its deſcent. Fig. VIII.

Let $GC = r$, $AC = a$, $AO = x$, V the velocity at O, then we have the force which accelerates the body at O $= F \times \frac{r^2}{\overline{a-x}^2}$, and if z be the ſpace due to the velocity at O, from the acceleration of the force 1,* then will $\dot{z} = \frac{Fr^2\dot{x}}{\overline{a-x}^2}$, and $z = \frac{Fr^2}{a-x}$, which ſhould vaniſh when $x = 0$, but when $x = 0$, $\frac{Fr^2}{a-x} = \frac{Fr^2}{a}$. Wherefore the entire fluent is $z = \frac{Fr^2}{a-x} - \frac{Fr^2}{a} = \frac{Fr^2x}{a \times \overline{a-x}}$, and the velocity in O, or $V = \sqrt{\frac{Fr^2x}{a \times \overline{a-x}}}$ in general, that is, when the ſtandards of ſpace, time, velocity, and force are each aſſumed equal 1: if the force F be referred to that which accelerates bodies through a ſpace $= l$, in the time 1,† we have $V = \sqrt{\frac{4lr^2Fx}{a \times \overline{a-x}}}$ expreſſed in the dimenſions of a ſtandard ſpace l, and time 1.

* Sect. III. Prop. V. Cor. 4.

† Sect. III. Prop. V. p. 28.

Fig. IX.

Cor. 1. Let DGB repreſent the ſurface, C the centre of the earth, AO any ſpace through which a body can actually deſcend: then referring to the preceding ſolution, we have $AC = a$, $GC = r$, $F = 1$, being the force of gravity at the earth's ſurface: $AO = x$, and the velocity generated in the point O, will be $\sqrt{\frac{4lr^2x}{a \times \overline{a-x}}}$, but if the force of gravity were uniform at all diſtances, and $= 1$, the velocity generated would ‡ be $\sqrt{4lx}$, the difference of which velocities $= \sqrt{4lx} - \sqrt{\frac{4lr^2x}{a \times \overline{a-x}}}$, $= \sqrt{4lx} \times \overline{1 - \sqrt{\frac{r^2}{a \times \overline{a-x}}}}$. In order to reduce this, let $s = AG$ being the altitude or diſtance above the earth's ſurface from which the body begins to fall, then will $a = r + s$ and $\overline{aa - ax} = r^2 + 2rs + s^2 - rx - sx$, that is, ſince $s^2 - sx$ is incomparably ſmaller than $2rs - rx$, we ſhall

‡ Sect. III. Prop. V. Cor. 3.

have $a^2 - ax = r^2 + 2rs - rx$, and $\sqrt{a^2 - ax} = r + \frac{2rs - rx}{2r} = \frac{2r^2 + 2rs - rx}{2r}$; the difference of the velocities therefore will approximate to $\sqrt{4lx} \times \overline{1 - \frac{2r^2}{2r^2 + 2rs - rx}} = \sqrt{4lx} \times \frac{2rs - rx}{2r^2}$ or $\sqrt{4lx} \times \frac{\overline{2s - x}}{2r}$ as a limit, while x and s, become continually ſmaller: we have therefore the following conſtruction: In the earth's radius CG, take $GI = GA$, then if a body falling from A, deſcribes the ſpace AO, the difference of the velocities, or the error occaſioned by aſſuming Galilæo's Hypotheſis of the uniform gravity of the earth at all diſtances from the centre, will be that part of the whole velocity generated, which is expreſſed by the fraction $\frac{IO}{2CG}$, when x and s vaniſh in reſpect of r. Fig. IX.

Cor. 2. If the body deſcends to the earth's ſurface, that is, to the point at which the earth's gravity is aſſumed $= 1$, the error in the velocity above mentioned, will be the whole velocity generated $\times \frac{AG}{2CG}$, that is, the error cannot be to the whole velocity generated in a greater proportion than that of the altitude fallen from to the earth's diameter.

Since in general the velocity $= \sqrt{\frac{Fr^2x}{a \times \overline{a - x}}}$, the quantities F, r, and a being given, and x being the variable ſpace deſcribed, the velocities in different points of the ſame deſcent, will be in a direct ſubduplicate ratio of the ſpace fallen through, and the inverſe ſubduplicate ratio of the ſpace to be deſcribed before the body arrives at the centre.

III.

Let a body deſcend from reſt in a right line toward a centre c, the attractive force of which varies in an inverſe dupli- Fig. VIII.

cate proportion of the diſtance from C; alſo let the force at any given point G be F: it is required to aſſign the time wherein the body deſcribes any given ſpace AO from reſt.

Let the given diſtance $CG = r$, the force at $G = F$, being referred to any ſtandard force 1 : let $AO = x$, $AC = a$, and the force at O will be that part of the force at G, which is expreſſed by the fraction $\frac{r^2}{\overline{a-x}^2}$, and the

* Sect. IV. Prop. II. velocity* at O will be $\sqrt{\frac{Fr^2x}{a \times \overline{a-x}}}$. If therefore the time be denoted by T, we have $\dot{T} = \sqrt{\frac{a}{Fr^2}} \times \sqrt{\frac{\overline{a-x} \times \dot{x}}{x}}$, and the time of deſcribing AO, that is, $T = \sqrt{\frac{a}{Fr^2}} \times$ into the fluent of $\dot{x}\sqrt{\frac{a-x}{x}}$; which may be obtained from the following conſtruction.

Let $CA = a$, upon which deſcribe a ſemicircle APC,
Fig. X. the centre of which is D, and in the line AC aſſuming any point O, draw the right ſine OP, then if AO be put $= x$, the fluent of $\sqrt{\frac{a}{Fr^2}} \times \sqrt{\frac{\overline{a-x} \times \dot{x}}{x}} = \sqrt{\frac{a}{Fr^2}} \times \overline{\overset{\frown}{AP} + PO} =$ the time of the body's deſcribing AO
† Sect. III. Prop. V. from reſt, when the ſtandards † of ſpace, time, velocity, and force are each aſſumed = 1, which may be thus demonſtrated.

Draw the chords AP and CP; then during the time in which the body deſcribes the element of ſpace Oo, the right ſine OP will become op, and if Ap, Cp be drawn, ſince $CPA = CmA$ is a right angle by the properties of the circle, the cotemporary variation of AP will be mp, and the circular area ACP will be during the ſame time increaſed by the addition of $PCp = \frac{CP \times mp}{2}$, which will therefore be the fluxion of the area ACP, and this area will be the fluent whereof mCp is the fluxion.

Let the arc $AP = A$, and the ſine $OP = S$, then ſince $AO = x$, $AC = a$, we have the chord $AP = \sqrt{ax}$, and its fluxion $\frac{\dot{x}\sqrt{a}}{2\sqrt{x}} = mp$: Moreover, $CP = \sqrt{a^2 - ax}$, wherefore PCp, or $mCp = \frac{CP \times mp}{2} = \frac{\sqrt{a^2 - ax}}{2} \times \frac{\sqrt{a}\,\dot{x}}{2\sqrt{x}} = \frac{a}{4} \times \sqrt{\frac{a-x}{x}}\,\dot{x}$, the fluent whereof = the area CAP = ſector DPA + the triangle $CDP = \frac{A \times a}{4} + \frac{\sqrt{ax} \times \sqrt{a^2 - ax}}{4}$ (becauſe the perpendicular $DQ = \frac{AP}{2} = \frac{\sqrt{ax}}{2}$) wherefore the area $CAP = \frac{A \times a}{4} + \frac{a}{4} \times \sqrt{ax - xx}$, and ſince $\sqrt{ax - xx} = OP = S$, the area $CAP = \overline{A + S} \times \frac{a}{4}$ = the fluent of $\frac{a}{4} \times \sqrt{\frac{a-x}{x}} \times \dot{x}$, and the fluent of $\sqrt{\frac{a}{Fr^2}} \times \sqrt{\frac{a - x \times \dot{x}}{x}} = \sqrt{\frac{a}{Fr^2}} \times \overline{A + S}$ = the time of deſcribing the ſpace AO from reſt, when the ſtandards of ſpace, time, velocity and force are each aſſumed ‡ equal 1; but if the force be referred to gravity, the time to one ſecond, and the ſpace to l, which a body deſcribes from reſt by the acceleration of gravity for 1 ſecond, we ſhall have *the time required $= \sqrt{\frac{a}{4Fr^2l}} \times \overline{A + S}$, expreſſed in ſeconds.

Cor. 1. The time of deſcribing from quieſcence any ſpace AO, will be proportional to the †circular area ACP, or to the ſum of the ſine and arc, whereof the ſpace deſcribed is the verſed ſine, the radius being half the diſtance of the body when quieſcent from the centre of attraction.

Cor. 2. It may be remarked concerning this determination,

‡ Sect. III. p. 28.

* p. 28.

† Newt. Princip. Prop. XXXII. Vol. I.

nation, that it is immaterial in what meaſure the ſpaces a, r, and l are expreſſed, viz. whether in feet, inches, &c. of any nation, provided they are expreſſed in the ſame meaſure; for the fluent $\sqrt{\frac{a}{4Fr^2l}} \times \overline{A+S}$ will always be a number, that is, a fraction, the numerator of which contains $\frac{3}{2}$ dimenſions of a line, and the denominator the ſame dimenſions of a line, wherefore whatever meaſure thoſe ſpaces be expreſſed in, the value of the fraction muſt be unaltered.

Cor. 3. If the body deſcends from quieſcence till it arrives at the centre C, the ſpace $AO = AC$, or $a = x$, in which caſe $S = 0$, and $T = \sqrt{\frac{a}{4lr^2F}} \times$ the ſemicircumference of a circle, the radius of which $= \frac{a}{2}$, this ſemicircumference $= \frac{pa}{2}$, p denoting the number 3.14159 &c. that is, $T = \frac{p\,a^{\frac{3}{2}}}{4r \times \sqrt{Fl}}$ ſeconds.

Cor. 4. For example: If the earth be conſidered as an immoveable centre of attraction, and the moon were to deſcend toward it in a right line by the force of the earth's gravity, the laſt Cor. will give the time of deſcent; for here, ſuppoſing the earth's radius to be $= 3970$ miles, we have $l = \frac{16\frac{1}{12} \text{ feet}}{3970 \text{ miles}} = \frac{1}{1303311}$, $r = 1$, $F = 1$, $p = 3.14159$, and $a = 60r$, the time therefore of the deſcent would be $= \frac{pa^{\frac{3}{2}}}{4r \times \sqrt{Fl}} = \frac{60^{\frac{3}{2}} \times 3.14159}{4 \times \sqrt{\frac{1}{1303311}}} = 416716$ ſeconds, or about 4 days and 20 hours.

Cor. 5. Since * the periodic time of any of the celeſtial bodies which compoſe the ſolar ſyſtem $=$ to $4\sqrt{2} \times$ into the time of its rectilinear deſcent to the centre from reſt, by the action of the centripetal force, it follows, that the periodic time of any primary or ſecondary planet $P = \frac{\sqrt{2pa^3}}{r\sqrt{Fl}}$, where P denotes the periodic time of the planet, a its diſtance from the centre of force, F a number which

* Newt. Princip. Prop. XV. Vol. I. and Prop. XXXII. Excerpt. e Newton. Princip. p. 87.

which is to 1, as the force at any given diſtance from the centre to the force of the earth's gravity at the ſame diſtance from the earth's centre, and $r =$ the diſtance from the former centre at which the force $= F$. Moreover, $P^2 = \frac{2p^2a^3}{r^2Fl}$, and $F = \frac{2p^2a^3}{r^2P^2l}$, by which equation the abſolute quantity of the centripetal force, compared with that of gravity at the ſame diſtance, will be known from the obſervation of the revolving body's periodic time and diſtance from the centre of its orbit. Thus, if the ſun's parallax be 11″, his diſtance from the earth will equal 18751 of the earth's radii; let this $= a$, and $l = \frac{1}{1303311}$, being that part of the earth's radius which a heavy body deſcending from reſt deſcribes in one ſecond: ſince P, or the periodic time of the earth $= 365.256 \times 24 \times 3600$ ſeconds, we have the force of the ſun $F = \frac{2p^2a^3}{r^2P^2l} =$
$$\frac{2 \times \overline{3.14159}^2 \times \overline{18751}^3}{\overline{365.256 \times 24 \times 3600}^2 \times \frac{1}{1303311}} = 170313,$$
a number expreſſing the ratio of the ſun's abſolute force to that of the earth. Moreover, ſince the ſun's diameter would be 87.545 times greater than that of the earth, the ſun's parallax being 11″, we have the ratio of the ſun and earth's magnitudes equal to that of 670967 : 1; and the denſities being as the quantities of matter or abſolute forces directly and magnitudes inverſely, the denſity of the ſun will be to that of the earth as $\frac{170313}{670967}$ to 1, or as 1 to 3.939. The ſun's parallax is aſſumed 11″ by Newton,* according to the opinion at his time: later obſervations have ſhewn the parallax to be conſiderably leſs. Euler in his treatiſe concerning the Tides, ‡ makes it 10″; if this were the real parallax it would follow, that the ſun's diſtance from us is equal to 20626 radii of the earth, and by the preceding rule the ſun's abſolute force comes out $=$
$$\frac{2 \times \overline{3.14159}^2 \times \overline{20626}^3}{\overline{365.256 \times 24 \times 3600}^2 \times \frac{1}{1303311}}$$
$= 22667$. But from the obſervations which were taken of the tranſit of Venus, over the ſun's diſk in the year 1769, in different parts of the earth, it is now concluded that the ſun's parallax is 8″. 6. This being the true parallax, the ſun's abſolute force will appear to be to

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* Newt. Princip. Vol. III. Prop. VIII.

‡ Jeſuits' Newton, Vol. III. p. 291.

to that of the earth, as 356398 : 1.

Cor. 5. When the deſcending body deſcribes a ſpace x, which is very ſmall in compariſon with the diſtance AC, then ſince the time in general $= \sqrt{\frac{a}{4r^2Fl}} \times \overline{A+S}$, A and S at the very beginning of the deſcent will become equal to each other, each being $\sqrt{ax-xx}$; wherefore the time in the very beginning of the motion $= \sqrt{ax-xx} \times \sqrt{\frac{a}{r^2Fl}}$, and ſince the time of deſcribing a ſpace x with the conſtant force *$F = \sqrt{\frac{x}{Fl}}$, we have the difference ariſing from Galilæo's hypotheſis $= \sqrt{\frac{a}{Flr^2}} \times \sqrt{ax-xx} - \sqrt{\frac{x}{Fl}} = \sqrt{\frac{x}{Fl}} \times \overline{\sqrt{\frac{a^2-ax}{r^2}}-1}$, that is, ſince x is incomparably ſmaller than a, the difference of the times will approximate to $\sqrt{\frac{x}{Fl}} \times \overline{\frac{a-\frac{x}{2}}{r}-1}$, $= \sqrt{\frac{x}{Fl}} \times \frac{2a-2r-x}{2r}$. In order to reduce this let $s = AG$ being the altitude above the earth's ſurface from which a body begins to deſcend, then will $a = r+s$, and $\frac{2a-2r-x}{2r} = \frac{2s-x}{2r}$, and the force at G, that is F being $= 1$, the difference in time above deſcribed $= \sqrt{\frac{x}{l}} \times \frac{2s-x}{2r}$, which gives the following conſtruction. In the earth's radius CG, take $GI = GA$: then if a body falling from A deſcribes the ſpace AO, the difference in time or error, occaſioned by aſſuming Galilæo's Hypotheſis of the earth's uniform gravity at all diſtances, is that part of the whole time which is expreſſed by the fraction $\frac{IO}{2CG}$.

Fig. IX.

Cor. 6. If the body deſcends to the point G, where the

* Sect. III. Prop. IV.

the force of gravity $= 1$, the error in time will approximate to that part of the whole time which is expreſſed by the fraction $\frac{AG}{2CG}$, that is, the error in time will bear the ſame proportion to the whole, as that of the height fallen from to the earth's diameter, when the quantity s vaniſhes in reſpect of the earth's radius.

IV.

Let a body ſituated at A, be attracted toward a centre of force C, the quantity of attraction being in the inverſe ratio of the diſtances from C; it is required to aſcertain the velocity of a body which has arrived at any point O in its deſcent from A.

Fig. VIII.

Let $AC = a$, $AO = x$, $GG = r$, the force at the given diſtance $CG = F$, then the force at O will be denoted by $F \times \frac{r}{a-x}$, and if $z =$ the ſpace through which a body muſt fall by the acceleration of the conſtant force 1, to acquire the velocity in O, then the principles of acceleration *give us $\dot{z} = \frac{Fr\dot{x}}{a-x}$, and $z = Fr \times -\log. \overline{a-x}$, which ſhould vaniſh when $x = 0$; but when $x = 0$, $Fr \times -\log. \overline{a-x}$, will $= Fr \times -\log. a$, the whole fluent, therefore, will be $z = Fr \times -\log. \overline{a-x} + Fr \times \log. a$, or $z = Fr \times \log. \frac{a}{a-x}$, and the velocity, when F is referred to gravity $= \sqrt{4lFr \times \log. \frac{a}{a-x}}$.

Fig. XI.

Let AGH repreſent a circle deſcribed by a body retained in its orbit by a centripetal force C, which force varies in an inverſe ratio of the diſtance from C. It is required to aſſign how far a body muſt fall within the circle

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* Sect. III, Prop. V. Cor. 4.

circle from reſt at A, by the action of the variable force at C, ſo as to acquire the velocity equal to that with which the body revolves in the circle. Let $AC = a$, AO the diſtance required $= x$, and let the force at $A = 1$, then will the force at $O = \frac{a}{a-x}$, and z ſignifying as before, $\dot{z} = \frac{a\dot{x}}{a-x}$, and $z = a \times \log. \frac{a}{a-x}$; but the velocity in the circle = the velocity acquired by a body which has fallen through $\frac{AC}{2}$, by the action of the conſtant force 1, and will therefore be $= \sqrt{\frac{a}{2}}$, wherefore we have by the problem $a \times \log. \frac{a}{a-x} = \frac{a}{2}$, and log: $\frac{a}{a-x} = \frac{1}{2}$. Let $e = 2.7182818$ being the number whoſe hyperbolic log. $= 1$, therefore $\frac{a}{a-x} = e^{\frac{1}{2}}$, $a = ae^{\frac{1}{2}} - e^{\frac{1}{2}}x$, and $x = \frac{ae^{\frac{1}{2}} - a}{e^{\frac{1}{2}}} = a - \frac{a}{e^{\frac{1}{2}}} = a \times$ ·39347·

Fig. XI.

If a body be projected perpendicularly from the circumference with the ſame velocity with which it revolves in the circle, the height to which it will riſe may be alſo aſcertained. For in this caſe, putting AQ the required ſpace $= x$, and proceeding as before, we have $z = -a \times \log. \overline{a+x}$, which ſhould vaniſh together, but when $x = 0$, $z = -a \times \log. a$, and by the problem when $x = 0$, $z = \frac{a}{2}$, wherefore the entire fluent will be $z = \frac{a}{2} + a \times \log. a - a \times \log. \overline{a+x}$, that is, $z = \frac{a}{2} + a \times \log. \frac{a}{a+x}$: and ſince by the problem the body is ſuppoſed to continue aſcending till it loſes all velocity, it follows, that $z = \frac{a}{2} + a \times \log. \frac{a}{a+x} = 0$, we have therefore $\log. \frac{a}{a+x} = -\frac{1}{2}$; let e be the number whoſe hyperbolic log. $= 1$, wherefore $\frac{a}{a+x} = \frac{1}{e^{\frac{1}{2}}}$, and $ae^{\frac{1}{2}} = a + x$, and $x = a \times \overline{e^{\frac{1}{2}} - 1} = a \times .6487$.

From the ſame principles it alſo follows, that whatever be the velocity with which a body be projected directly from a centre of force, varying in the inverſe ratio of the diſtances, it will always return: for let y be the ſpace through which a body muſt deſcend from reſt by the acceleration of the force 1, ſo that it may acquire the velocity with which the body is projected from the point A, then we have from the preceding ſolution $z = - a \times \log. \overline{a + x}$, and the whole fluent $z = y + a \times \log. a - a \times \log. \overline{a + x}$, or $z = y + a \times \log. \frac{a}{a+x}$; and when the velocity of projection is deſtroyed or $z = 0$, the equation ~~of it~~ will be $a \times \log. \frac{a}{a+x} = -y$, or $a \times$ log. $\frac{a+x}{a} = y$, and x being infinite, y will be infinite alſo.

It appears therefore, that if the projected body returns not, the velocity of projection muſt have been infinite.

V.

Let a body ſituated at A, be repelled by a force acting from a centre C, always being in an inverſe ratio of the diſtances from C: moreover, let the force at any given point G = F, it is required to aſcertain the velocity of the body when it has arrived at O. Fig. XII.

Let $CG = r$, $AC = a$, $CO = x$, then by the problem the force at $O = F \times \frac{r}{x}$, and if z be the ſpace thro' which a body muſt fall freely by a ſtandard force 1, for example that of the earth's gravity, to acquire the velocity in O, we

* Sect. III. Prop. V. Cor. 4.

‡ Sect. III. p. 28.

have * $\dot{z} = \frac{Fr\dot{x}}{x}$, and $z = Fr \times \log.\ x$, when therefore $x = a$, $z = Fr \times \log.\ a$, but by the problem when $x = a$, $z = 0$, and the entire fluents will be $z = Fr \times \log.\ \frac{x}{a}$, and the velocity ‡ acquired in $O = \sqrt{4Frl \times \log.\ \frac{x}{a}}$.

VI.

Every thing remaining as in the laſt propoſition, let it be required to determine the time wherein the body deſcribes any given ſpace A O from reſt.

† Sect. IV. Prop. V.

§ Sect. III. Prop. III.

Since the velocity † at $O = \sqrt{4rlF \times \log.\ \frac{x}{a}}$, if T be the time of deſcribing AO, we have § $\dot{T} = \frac{1}{\sqrt{4rlF}} \times \frac{\dot{x}}{\sqrt{\log.\ \frac{x}{a}}}$, and the time required $T = \frac{1}{\sqrt{4rlF}} \times$ fluent of $\frac{\dot{x}}{\sqrt{\log.\ \frac{x}{a}}}$. In order to obtain the fluent, let $\log.\ \frac{x}{a} = v$: if e be the number 2.7182818, the hyperbolic logarithm of which $= 1$, then will $\frac{x}{a} = e^{v}$, and $x = ae^{v}$: alſo ſince $\log.\ \frac{x}{a} = v$, taking the fluxions $\frac{\dot{x}}{x} = \dot{v}$, and $\dot{x} = x\dot{v} = a\dot{v}e^{v}$; wherefore $\frac{\dot{x}}{\sqrt{\log.\ \frac{x}{a}}} = \frac{a\dot{v}e^{v}}{\sqrt{v}}$; but $e^{v} = 1 + \frac{v}{1} + \frac{v^2}{1.2} + \frac{v^3}{1.2.3} + \frac{v^4}{1.2.3.4}$ &c. and

a

$$\frac{a\dot{v}e^{v}}{\sqrt{v}} = a v^{-\frac{1}{2}}\dot{v} + \frac{a v^{\frac{1}{2}}\dot{v}}{1} + \frac{a v^{\frac{3}{2}}\dot{v}}{1.2} + \frac{a v^{\frac{5}{2}}\dot{v}}{1.2.3} +$$

$\frac{a v^{\frac{7}{2}}\dot{v}}{1.2.3.4}$, &c, wherefore the fluent of $\frac{\dot{x}}{\sqrt{\log.\frac{x}{a}}}$ = the fluent of $\frac{a\dot{v}e^{v}}{\sqrt{v}} = 2av^{\frac{1}{2}} + \frac{2av^{\frac{3}{2}}}{3.1} + \frac{2av^{\frac{5}{2}}}{5.1.2} + \frac{2av^{\frac{7}{2}}}{7.1.2.3}$, &c. and the time of deſcribing AO required $= \sqrt{\frac{1}{4rlF}} \times$ fluent of $\frac{\dot{x}}{\sqrt{\log.\frac{x}{a}}} = \sqrt{\frac{a^2 v}{rFl}} \times \overline{1 + \frac{v}{3.1} + \frac{v^2}{5.1.2} +}$ $\overline{\frac{v^3}{7.1.2.3}}$, &c.

From the laſt two propoſitions we might derive an explanation of the laws according to which motion is communicated to bodies, by the elaſtic force of the air and other ſimilar ſubſtances; but the particular conſtruction of the inſtruments, by means of which the elaſtic fluid and vapour poſſeſſing ſimilar powers of elaſticity are applied to communicate motion, require that theſe ſubjects ſhould be ſeparately conſidered.

The air is a fluid, poſſeſſing a property which diſcriminates it from moſt other fluids; thus, a quantity of water, mercury, &c. of any given magnitude, can never be reduced in its dimenſions ſo as to occupy a ſmaller portion of ſpace; whereas the air by compreſſion ſuffers ſuch diminution of its bulk as is in proportion to the compreſſion, that is, the ſpace occupied by it is always leſs as the compreſſive force is greater. Now ſince action and reaction are equal and in contrary directions, it is plain, that whatever force be neceſſary to reduce the air into dimenſions leſs than thoſe which it naturally poſſeſſes in the atmoſphere,[*] the force with which the air endeavours to reſtore itſelf muſt be equal to that of compreſſion: ſo that if a quantity of air occupies 1000 times leſs ſpace than when it is diffuſed in the atmoſphere, it will exert an expanſive force 1000 times greater than that with which it reacts againſt the air ſurrounding it in a natural ſtate: and this conveys an adequate idea of the air's

[*] Cotes' Hydroſtat. Lect. IX.

elaſtic force: what the quantity of this force is will be made eaſily to appear; for the elaſticity of the air near the ſurface of the earth being equal to the force which compreſſes it, we ſhall have the quantity of the elaſtic force exerted on each ſquare inch of ſurface, equal to the weight of the atmoſphere preſſing upon the ſame ſurface, that is, about 15 pounds avoirdupoiſe. If therefore 1000 cubic inches of air be included in the magnitude of one inch, this included air will exert a force of elaſticity upon each ſurface of the cube equal to the weight of 15000 pounds avoirdupoiſe. Let us imagine a perforation to be made in one of the ſides of this cube: if a body, for example a leaden ball, be placed cloſe to the aperture, the expanſive force of the air would propel it; in this caſe however the air having room to expand itſelf in all directions, the whole force of it would not be employed in communicating motion to the ball: but if a cylindrical tube, equal in diameter to the ball, be fitted cloſe to the aperture, and the ball be placed as before, the air now having no other paſſage or means of the expanſion but by the protuſion of the ball, will be wholly employed in communicating motion to it.

To find the velocity of this motion communicated in given circumſtances, it is to be firſt conſidered, that if the force of the air's expanſion continued to act uniformly during the whole progreſs of the ball through the barrel, the velocity communicated would eaſily follow from the principles of conſtant acceleration: let the length of the barrel be s, the force of the air's expanſion $n = 1000$ times the ball's weight, $l = 16\frac{1}{12}$ feet, v the velocity wherewith the ball would in that caſe quit the barrel, then we ſhould have * $v = \sqrt{4lns}$ feet in a ſecond; but this velocity is manifeſtly greater than that which is in reality communicated; for although the firſt effort of the air compreſſed into $\frac{1}{1000}$ of its natural ſpace, is equivalent to a weight of 15000 pounds upon every ſquare inch of ſurface preſſed on, yet after it has by propelling the ball expanded itſelf into larger dimenſions, the force whereby it accelerates the ball's motion is proportionally diminiſhed; when therefore the cubic inch of compreſſed air has expanded itſelf, ſo as to occupy a ſpace of two cubic inches, the force of expanſion will be only one half of what it was before, that is, a force of 7500 upon each ſquare inch of ſurface, and in proportion for leſſer or greater ſurfaces.

* Sect. III. Prop. V. Cor. 3.

The elaſtice force of the air will therefore continually be diminiſhed as the ball proceeds along the barrel, and of courſe the increment of velocity generated in a given element of time diminiſhed proportionally: the principles contained in the preceding propoſitions will be ſufficient from having given the quantity of compreſſed air and the denſity of it, together with the dimenſions of the ball and cylinder to deduce the velocity wherewith the ball quits the cylinder, and the time of motion during its acceleration.

VII.

Let B D repreſent a cavity which contains compreſſed air, to this cavity let a cylinder A Q C be fixed, and let a leaden ball be applied to A, oppoſite to a cloſed aperture of a diameter equal to that of the ball or cylinder: the aperture being ſuddenly opened the ball will ruſh forth from the barrel; it is required from the neceſſary data to determine the velocity with which the ball iſſues from C. Fig. XIII.

Suppoſe the compreſſed air to occupy $\frac{1}{m}$ parts of its natural bulk in the atmoſphere, ſo that the elaſtic force of it exerted againſt any ſurface, may be m times the weight of the atmoſphere preſſing againſt the ſame ſurface: moreover, let the cavity contain c cubic inches of this compreſſed air, and let the diameter of the ball or cylinder $= d$, its weight $= w$, the length of the cylinder $= AC = s$, $p = 3.14159$, and then will $\frac{d^2 p}{4}$ be the area of a circle, the plane of which biſects the ball, and cuts the cylinder in a direction perpendicular to the axis: and ſince the air in its natural ſtate preſſes on a ſquare inch of ſurface with a force equivalent to the weight of 15 pound or 240 ounces avoirdupoiſe, it follows, that the elaſtic force of air, if it be compreſſed into m times leſs ſpace than that which it occupies in the atmoſphere, acting againſt the ſurface $\frac{d^2 p}{4}$ will be $\frac{240\, m d^2 p}{4}$ ounces, which will be the abſolute or moving force of the compreſſed air impelling the ball at A, and if the weight of the ball be w, the accelerating force which generates velocity in the ball at A, will be $\frac{240\, m d^2 p}{4 w} = \frac{60\, m d^2 p}{w}$.

Now, ſuppoſe the ball to have been advanced to Q, and let $AQ = x$, then will the cylindrical ſpace $AQ = \frac{d^2 p x}{4}$; and ſince the air before its expanſion occupied a ſpace $= c$ cubic inches, when the ball arrives at Q, it will occupy $c + \frac{d^2 p x}{4}$ cubic inches, the elaſtic force therefore of the air's expanſion at Q will be $\frac{60\, m d^2 p}{w} \times \frac{4 c}{4 c + d^2 p x} = \frac{240\, m d^2 p c}{4 c w + d^2 p w x}$.

Let z be the ſpace through which a body muſt fall from reſt by the acceleration of gravity, to acquire the velocity of the ball at Q, ‡ then we have $\dot{z} = \frac{240\, m d^2 p c \dot{x}}{4 c w + d^2 p w x}$, and taking the fluents $z = \frac{240\, m c}{w} \times \log. \overline{\frac{4 c + d^2 p x}{4 c}}$; that is, when $x = s$, or when the ball has arrived at the extremity of the cylinder, $z = \frac{240\, m c}{w} \times \log. \overline{\frac{4 c + d^2 p s}{4 c}}$.

Let A be the capacity of the veſſel containing the compreſſed air; alſo let B be the capacity of the barrel or cylinder, then will the quantity $\frac{4 c + d^2 p s}{4 c} = \frac{A + B}{A}$, and we ſhall have $z = \frac{240\, m A}{4} \times \log. \frac{A + B}{A}$, and the velocity

‡ Sect. III. Prop. V. Cor. 4.

velocity* required $V=\sqrt{\frac{4\times193\times240\,Am}{w}}\times\sqrt{\log.\frac{A+B}{A}}$, * Sect. III. p. 28.

that is, $V=\sqrt{\frac{185280\,mA}{w}}\times\sqrt{\log.\frac{A+B}{A}}$ inches in a ſecond.

To find the time of the ball's motion through any ſpace $AO=x$, let T be the time required, wherefore † $\dot{T}=$ † Prop. III. Prop. III.

$\sqrt{\frac{w}{185280\,mA}}\times\frac{\dot{x}}{\sqrt{\log.\frac{4A+d^2px}{4A}}}$; then if $v=$ log. of $\frac{A+B}{A}$ the fluent of the equation is obtained from the preceding ſolution, § and we ſhall have the time of deſcribing the ſpace $AO=\sqrt{\frac{wAv}{2895\,d^4p^2m}}\times\overline{1+\frac{v}{3.1}+\frac{v^2}{5.1.2}+\frac{v^3}{7.1.2.3}}$, &c. a few terms of which will approximate to the true time with ſufficient exactneſs, if the capacity of the barrel does not exceed the capacity of the veſſel which contains the compreſſed air, ſo that the logarithm of the quantity $\frac{A+B}{A}$ ſhall not exceed .692; but in the uſual conſtruction of the air gun, the quantity $\frac{A+B}{A}$ rarely exceeds $\frac{11}{10}$, and conſequently its logarithm will be leſs than .133; in which caſe three terms of the preceding ſeries will give the time true to the ſixth decimal place. The logarithms uſed in this as well as in the preceding and ſubſequent ſolutions are always underſtood to be hyperbolic, unleſs the contrary be expreſſed. § Sect. IV. Prop. VI.

Cor. 1. If the capacity of the barrel and of the veſſel containing compreſſed air remain the ſame, the velocity of the ball will depend only on its weight and the air's compreſſion, and not in the length of the barrel.

Cor. 2. Every thing elſe being given, the ball's velocity will be in an inverſe ſubduplicate ratio of its ſpecific gravity: thus, balls of iron and lead diſcharged from the ſame air gun by the elaſtic force of air equally compreſſed and equal in quantity, will iſſue forth with velocities which are in the proportion of $\sqrt{11.3}$, $\sqrt{7.3}$, or as 1.2 : 1.

Cor. 3. If the ſpecific gravities of two balls be the ſame, the velocities wherewith they quit the air gun, will be in a direct ſubduplicate ratio of the air's compreſſion: thus, if two leaden balls be diſcharged from the ſame air gun, and the air being compreſſed in one caſe 32 times, and in the other 8 times, the velocities of the ball's egreſs will be as 2 : 1.

Cor. 4. The theory will be further illuſtrated by referring to an air gun of the uſual dimenſions, and from the data belonging to the problem aſcertaining the actual velocity communicated to the ball, and the time of its acceleration.

Suppoſe the cavity containing compreſſed air to be a ſphere of 4 inches in diameter, and conſequently equal in capacity to 33.510 cubic inches: ſuppoſe the ball to be of lead, and in diameter equal .4 parts of an inch, wherefore the ſpecific gravity of lead being 11.3, the ball's weight = .21913 of an avoirdupoiſe ounce. Moreover, let the air be compreſſed into 20 times a ſmaller ſpace than that which it occupies in the atmoſphere, and let the length of the barrel $= s$, we have therefore for the ſolution the following data, $A = 33.51$, $m = 20.00$, $d = .40$, $p = 3.14159$, &c. $w = .21913$, $s = 42.00$ inches; wherefore the capacity of the barrel $= \frac{d^2ps}{4} = 5.2778 = B$: we have alſo $\frac{A+B}{A} = \frac{38.7878}{33.51}$, the logarithm of which from the tables $= .06352$, which being multiplied by 2.30258, will become the hyperbolic logarithm of $\frac{A+B}{A} = .14626$, and the ball's velocity at $C = \sqrt{\frac{4 \times 193 \times 240\,mA}{w}} \times \sqrt{\log. \frac{A+B}{A}} = \sqrt{\frac{4 \times 193 \times 240 \times 20 \times 33.51}{.21913}} \times \sqrt{.14626} = 9103.9$ inches, or 758.6 feet in a ſecond.

In order to obtain the time of the ball's motion through the cylinder, we have v = the log. of $\frac{A+B}{A} = .14626$, and the other notation remaining

$\surd$

$$\sqrt{\frac{wAv}{2895\,d^4p^2m}} = \sqrt{\frac{.21913 \times 33.51 \times .14626}{2895 \times .4^4 \times \overline{3.14159}^2 \times 20}} =$$

.0085683, and $\overline{1 + \frac{v}{3.1} + \frac{v^2}{5.1.2}} = 1.0508$, and the time required $= \sqrt{\frac{wAv}{2895\,d^4p^2m}} \times \overline{1 + \frac{v}{3.1} + \frac{v^2}{5.1.2}}$, &c. $=$

$.0085683 \times 1.0508 = .00900 = \frac{1}{111}$ part of a ſecond.

In this problem the air's condenſation has been aſſumed as great as can be uſed conveniently in phyſical experiments, but far ſhort of that which exiſts in ſome ſubſtances both natural and artificial:, to mention a ſingle inſtance only: it is well known that the elaſtic ſteam of gunpowder, which remains in a fixed ſtate till the inſtant of exploſion, poſſeſſes the properties of air, condenſed in about 1000 times a leſs ſpace than that which it occupies in the atmoſphere. Three ‡ parts in ten of the gunpowder's weight conſiſt of this condenſed air or vapour, which being ſet at liberty by the application of fire, will exert a force of preſſure equivalent to a thouſand times the weight of the atmoſphere preſſing againſt the ſame ſurface, from which cauſe ariſes the great power of this ſubſtance in communicating motion to ponderous bodies. This elaſtic ſteam like the air expands itſelf with a force inverſely proportional to the ſpace which it occupies, and conſequently the preceding ſolution is applicable to the action of gunpowder as well as to that of air abſolutely compreſſed, and will in both caſes be ſufficiently accurate, provided the velocity communicated to the ball bears a very ſmall proportion to the velocity with which the air or elaſtic vapour would expand itſelf if not impeded: although the impelling force of the elaſtic ſteam or compreſſed air has been aſſumed equal to the force of its preſſure acting againſt a quieſcent ball, yet this is ſtrictly true only at the firſt inſtant the ball begins to be moved; afterwards the magnitude of the impulſe ſhould be eſtimated by the difference between the velocity with which the air or ſteam would expand itſelf if no ways impeded, and the ball's velocity: this conſideration is neceſſary to render the ſolution general: if the accelerating force be aſſumed equal to the air's preſſure divided by the ball's weight, as it is in this as well as in † Mr. Robins' ſolution,

‡ Robins' Gunnery, Vol.I. p.73.

† Robins' Gunnery, Vol.I. p.75.

and we diminiſh the ball's weight ſine limite, the accelerating force will be infinite, and conſequently the velocity generated greater than any that is aſſignable; whereas it is impoſſible that the velocity communicated can in any caſe exceed that with which the fluid would expand itſelf if not at all impeded: this circumſtance therefore muſt in ſome degree affect the truth of the ſolution, (notwithſtanding the great denſity of the ball,) and ſhould be allowed for accordingly whenever great accuracy is required. In the mean time it may be remarked, that the velocity with which the elaſtic vapour of gunpowder will expand itſelf, the obſtacles which confine it being removed, is not ſufficiently aſcertained: Mr. Robins eſtimates it at about 2000 feet in a ſecond, though ſome other ‡ authors make it five times greater. It is a ſubject which ſcarcely admits of theoretical computation; and great difficulties occur in making experiments relative to it. If the denſity of the vapour of gunpowder in its compreſſed ſtate were the ſame as that of atmoſpheric air, its claſticity being 1000 times greater, and its progreſs was not impeded by the groſs matter which enters into the compoſition of gunpowder nor by any other obſtacle, the velocity with which the compreſſed vapour expands itſelf would be 40900 feet in a ſecond.

‡ D. Bernouilli.

From the near agreement between the experiments of Mr. Robins on the actual velocities of muſket balls, and the deductions from his theory, it may be concluded that the velocity with which the vapour of gunpowder expands itſelf, when not impeded, is much greater than that of 2000 feet in a ſecond.

VIII.

Fig. VIII.

Let a body A be attracted toward a point or centre C, with forces which vary in that power of the diſtance from C, which is expreſſed by the general index n. Alſo, let the force at a given diſtance C G be = F; it is required to aſſign the velocity acquired by the body in deſcribing any ſpace A O from reſt by the attraction of the variable force.

Let $CG = r$, $AC = a$, $AO = x$, $Oo = \dot{x}$, V the velocity of the body while it is deſcribing Oo: then will the force acting on the body at $O = \frac{F \times \overline{a-x}^{n}}{r^{n}}$, and if z be the ſpace through which a body muſt move from reſt by the action of a ſtandard force 1,* in order to generate the velocity V, the principles of acceleration give us $\dot{z} = \frac{F \times \overline{a-x}^{n}}{r^{n}}\dot{x}$, and taking the fluent, $z = \frac{-F \times \overline{a-x}^{n+1}}{\overline{n+1}\, r^{n}}$, this being the variable part of the fluent which ſhould vaniſh when $x = 0$; but in that caſe $\frac{-F \times \overline{a-x}^{n+1}}{\overline{n+1}\, r^{n}} = \frac{-F a^{r+1}}{\overline{n+1}\, r^{n}}$, the conſtant quantity therefore $\frac{F a^{n+1}}{\overline{n+1} \times r^{n}}$ being added to $\frac{-F \times \overline{a-x}^{n+1}}{\overline{n+1}\, r^{n}}$, we have $z = \frac{F \times \overline{a^{n+1} - \overline{a-x}^{n+1}}}{\overline{n+1}\, r^{n}}$ = the entire fluent, and the velocity required in terms of the general ſtandard 1, will be $V = \sqrt{\frac{F \times a^{n+1} - \overline{a-x}^{n+1}}{\overline{n+1} \times r^{n}}}$.

* Sect. III. Prop. V. Cor. 4.

Cor. 1. If $n = -1$, the expreſſion for the velocity V fails: in this caſe the ſolution is obtained otherwiſe from Prop. IV.

Cor. 2. When $a = x$, and $n + 1$ is poſitive; the whole ſpace to the centre being deſcribed, the velocity $V = \sqrt{\frac{F \times a^{n+1}}{\overline{n+1} \times r^{n}}}$.

Cor. 3. When $n+1$ is negative, let $n+1 = -m$, and the general equation for the velocity acquired will become $V = \sqrt{\frac{F \times \overline{a^{-m} - \overline{a-x}^{-m}}}{-mr^{-m-1}}} = \sqrt{\frac{F \times \overline{\overline{a-x}^{m} - a^{m}}}{-ma^{m} \times \overline{a-x}^{m} \times r^{-1-m}}}$, and when $x = a$, that is, the whole ſpace to the centre being deſcribed, the velocity acquired at C, or $\sqrt{\frac{F \times \overline{\overline{a-x}^{m} - a^{m}}}{-ma^{m} \times \overline{a-x}^{m} \times r^{-1-m}}}$ will be infinite. If therefore the force varies in any inverſe ratio of the diſtances from the centre equal to or greater than that which is expounded by -1, the velocity acquired by a body which has deſcribed from reſt any ſpace AC to the centre will be infinite.

Cor. 4. Let the whole diſtance $AC = A$, and $OC = P$; moreover, let $n+1 = m$, then if a body deſcends from A towards C, the velocity acquired at any point † O will $= \sqrt{\frac{\overline{A^m - P^m} \times F}{mr^{m-1}}}$, and $F \times \frac{1}{mr^{m-1}}$ being given, the velocity will be proportional to* $\sqrt{A^m - P^m}$.

† Vid. ſupra p. 83.

* Newt. Princip. Vol. I. Prop. XL. Cor. 2.

Cor. 5. Since ‡ $V = \sqrt{\frac{F \times \overline{a^{n+1} - \overline{a-x}|^{n+1}}}{\overline{n+1}\, r^{n}}}$, if the time of deſcribing AO be T, we have the time of deſcribing the ſpace Oo or $\dot{T} = \frac{\sqrt{r^{n} \times \overline{n+1} \times \dot{x}}}{\sqrt{F \times \overline{a^{n+1} - \overline{a-x}|^{n+1}}}}$, and the time $T =$ the fluent of $\frac{\sqrt{r^{n} \times \overline{n+1} \times \dot{x}}}{\sqrt{F \times \overline{a^{n+1} - \overline{a-x}|^{n+1}}}}$, which muſt be determined for each particular value of n.

‡ Supra p. 83.

IX.

Fig. XIV. Let a body ſituated at A, be attracted toward a centre C by a ſtandard force 1; let AC be biſected in G, and ſuppoſing the

force which tends to C, to vary in any ratio of the diſtance, which is expounded by the general index n; it is required to aſſign through what ſpace AO the body muſt move from reſt at A by the attraction of the variable force, ſo that the velocity acquired ſhall be equal to that which would be generated in a body, deſcending from reſt through the ſpace AG by the action of the ſtandard force 1 continued uniform.

Let $AC = a$, and $AG = \frac{a}{2}$; moreover, let $AO = x$, and let z be the ſpace through which a body muſt move from reſt by the acceleration of the conſtant force 1, to acquire the velocity in O. Then while the body is deſcribing the element of ſpace $Oo = \dot{x}$, the force acting on it will be $\frac{\overline{a-x}^{n}}{a^{n}}$, and by the principles of acceleration * we have $\dot{z} = \frac{\overline{a-x}^{n}\,\dot{x}}{a^{n}}$, and taking the fluents $z = \frac{-\overline{a-x}^{n+1}}{\overline{n+1}\,a^{n}}$, which is the variable part of the fluent only, and ſince when $x = 0$, $\frac{-\overline{a-x}^{n+1}}{\overline{n+1}\,a^{n}} = \frac{-a}{n+1}$, the whole fluent will be $\frac{a}{n+1} - \frac{\overline{a-x}^{n+1}}{\overline{n+1}\,a^{n}}$ = to the ſpace through which a body muſt deſcend from A by the action of the uniform force 1, ſo that the velocity may be equal to that which the body in O acquires by the action of the variable force, while it deſcribes AO: we have therefore from the problem $\frac{a}{n+1} - \frac{\overline{a-x}^{n+1}}{\overline{n+1}\,a^{n}} = \frac{a}{2}$, and $\frac{\overline{a-x}^{n+1}}{a^{n}} = a - \frac{na+a}{2} =$

* Sect. III. Prop. V. Cor. 4.

a

$\frac{a-na}{2}$; wherefore $\overline{a-x}^{n+1} = a^{n+1} \times \frac{1-n}{2}$, and

$$a - x = a \times \overline{\frac{a-n}{2}}\Big|^{\frac{1}{1+n}}, \text{ and } x = a - a \times \overline{\frac{1-n}{2}}\Big|^{\frac{1}{1+n}}.$$

Cor. 1. When a body revolves in a circle, being retained in its orbit by a centripetal force at C, the velocity with which the body revolves is equal to that which a body would acquire by falling from reſt, ‡ through half the radius, by the action of a conſtant force equal to the centripetal force acting on the body at the circumference of the circle which it deſcribes. It appears, therefore, from the ſolution, that when a body revolves in a circle, being retained in the circumference by the action of a centripetal force, the variation of which is expounded by any general power of the diſtance n, the ſpace through which a body muſt be attracted from the circumference by the variable centripetal force, ſo that it may acquire the velocity in the circumference is $AO = a - a\,\overline{\frac{1-n}{2}}\Big|^{\frac{1}{1+n}}$, a being the radius of the circle.

‡ Excerpt. e Newton. Princip. p. 38.

Cor. 2. When $n = 0$, the ſpace $AO = a - \frac{a}{2} = \frac{a}{2}$.

Cor. 3. If $n = 1$, that is, if the force is as the diſtance $\overline{\frac{1-n}{2}}\Big|^{\frac{1}{1+n}} = 0$, and the ſpace $AO = a - a\,\overline{\frac{1-n}{2}}\Big|^{\frac{1}{1+n}} = a$, that is, the body muſt deſcend through the whole diſtance to the centre.

Cor. 4. If $n = -2$, $a \times \overline{\frac{1-n}{2}}\Big|^{\frac{1}{1+n}} = a \times \overline{\frac{3}{2}}\Big|^{-1} = a \times \overline{\frac{2}{3}}\Big|$; and the ſpace AO, through which a body muſt deſcend toward the centre to acquire the velocity in the circle $= a - a \times \frac{2}{3}$, that is, it muſt deſcend through $\frac{1}{3}$ radius.

Cor. 5. When n is $= -1$, $AO = a - a \times \overline{\frac{2}{2}}\Big|^{\frac{1}{0}}$ which being a *logarithm is not reducible to the preſent ſolution.

* Sect. IV, Prop. IV.

Cor. 6. Alſo, if n is poſitive and greater than 1, the problem is impoſſible, but with ſome circumſtances which become remarkable; for in this caſe $\overline{1-n}$ muſt be always negative, and if n is an odd number, $n+1$ will be an even number, and the quantity $\overline{\frac{1-n}{2}}\Big|^{\frac{1}{1+n}}$ will be impoſſible, as no root of a negative number can be extracted, when the index is the reciprocal of an even number; but if n be an even number, $n+1$ will be an odd number, and the root $\overline{\frac{1-n}{2}}\Big|^{\frac{1}{n+1}}$ will be obtainable in rational terms, but the anſwer will be inconſiſtent with the conditions of the problem; for in this caſe the root $\overline{\frac{1-n}{2}}\Big|^{\frac{1}{1+n}}$ being neceſſarily negative, the quantity $-a \times \overline{\frac{1-n}{2}}\Big|^{\frac{1}{1+n}}$ will be poſitive, and the ſpace AO intercepted between the circumference and centre will become $a +$ the root before deſcribed, a quantity greater than radius, which is impoſſible. It follows, however, when the force varies in a direct ratio of the diſtance which is expounded by any index greater than 1, that if a body revolves in a circle by the action of the force, it cannot fall from reſt through any part of the radius by the attraction of the variable centripetal force, ſo as to acquire the velocity with which it revolves in the circle.

X.

Fig. XV.

Let BAD be a circle wherein a body revolves, being retained in its orbit by a centripetal force at C; let the force at the circumference A be denoted by 1, and ſuppoſing the variation of the force at other diſtances to be in that power of the diſtances, which is expreſſed by the general index n: if a body be projected from the circumference at A, and in a direction perpendicular to it with the ſame velocity wherewith the body revolves; it is required to aſſign the greateſt diſtance to which it will aſcend.

Let $AC = r$, the diſtance required $AO = x$, and the force at A being $= 1$, the force at any point O will be $\frac{\overline{r+x}^{n}}{rn}$, and if z be a ſpace through which a body muſt move from reſt by the conſtant acceleration of the force 1, to acquire the velocity in O, we * have $\dot{z} = -\frac{\overline{r+x}^{n}}{r^{n}} \times \dot{x}$, and taking the fluent $z = -\frac{\overline{r+x}^{n+1}}{r^{n} \times \overline{n+1}}$, being the variable part of the fluent only, and requiring a conſtant quantity to be added to it.

What this conſtant quantity is appears from the problem, for when $x = 0$, $z =$ the ſpace due to the velocity of projection $= \frac{r}{2}$; but by the preceding equation, when $x = 0$, $z = -\frac{r}{n+1}$ the whole fluent therefore will be

* Sect. III. Prop. V. Cor. 4.

r

$\frac{r}{n+1} + \frac{r}{2} - \frac{\overline{r+x}^{n+1}}{n+1 \times r^n} = z$: but as the body is ſuppoſed by the problem to aſcend till its velocity $= 0$, it follows, that $z = 0 = \frac{r}{n+1} + \frac{r}{2} - \frac{\overline{r+x}^{n+1}}{n+1 \times r^n}$; wherefore $\frac{\overline{r+x}^{n+1}}{r^n} = \frac{\overline{n+3} \times r}{2}$, and $\overline{r+x}^{n+1} = \frac{\overline{n+3} \times r^{n+1}}{2}$, $r + x = \left.\frac{n+3}{2}\right|^{\frac{1}{n+1}} \times r$, and the diſtance AO to which the body will aſcend $= x = r \times \left.\frac{n+3}{2}\right|^{\frac{1}{n+1}} - r$.

Cor. 1. If $n = 1$, that is, if the force be directly as the diſtance from C, the whole ſpace AO to which the body will aſcend, when projected with a velocity equal to that with which it revolves in the circle, will be $= r \times \sqrt{2} - r = r \times \overline{\sqrt{2} - 1}$.

Cor. 2. If the force be inverſely as the ſquare of the diſtance, that is, if $n = -2$, then the body projected in the manner above deſcribed will riſe to an height $= 2r - r = r$, that is, to an height equal to the radius.

Cor. 3. If the force varies in an inverſe duplicate ratio of the diſtance, a body having deſcended from reſt through the ſpace OA equal to the radius, will, when it arrives at the circumference, have acquired a velocity equal to that in the circle, becauſe the ſame motion will be acquired by acceleration in this caſe, as was loſt by retardation in the laſt.

Cor. 4. Let y be the ſpace through which a body muſt move from reſt by the acceleration of the force at A or 1, ſo as acquire the velocity of projection: we have from the ſolution, $\frac{r}{n+1} + y - \frac{\overline{r+x}^{n+1}}{n+1 \times r^n} = z$, being the ſpace due to the velocity at O when $AO = x$: if then z be made $= 0$, and x increaſed ſine limite, the ultimate value of y will give the velocity with which a body projected ed from any point A will never return: we have therefore the equation $\frac{r}{n+1} + y - \frac{\overline{r+x}^{n+1}}{\overline{n+1} \times r^n} = 0$, or $y = \frac{\overline{r+x}^{n+1}}{\overline{n+1} \times r^n} - \frac{r}{n+1}$.

It appears from this equation, that if n be any poſitive number, or any negative proper fraction, the velocity with which a body muſt be projected ſo as not to return, is greater than any finite velocity; becauſe $\frac{\overline{r+x}^{n+1}}{\overline{n+1} \times r^n}$ in this caſe is infinite.

Cor. 5. If $n = -1$, this ſolution fails; but the ſolution is obtained from Prop. IV.

Cor. 6. If n be any negative number greater than unity, $\frac{\overline{r+x}^{n+1}}{\overline{n+1} \times r^n}$ becomes evaneſcent, and $y = -\frac{r}{n+1}$: thus if $n = -2$, we have $y = \frac{-r}{n+1} = \frac{-r}{-2+1} = r$, that is, if the force varies in the inverſe duplicate ratio of the diſtances from the centre, and a body be projected from A in the direction AO, with a velocity which would be acquired by a body deſcending from reſt through a ſpace equal to the radius, by the acceleration of a conſtant force equal to that which acts on the body at the point A, the body thus projected will never return. Thus, if a body were projected from the earth's ſurface in a direction perpendicular to the horizon, with a velocity equal to that which a body would acquire in deſcending from reſt through the earth's radius, by the acceleration of a force conſtantly equal to the earth's gravity at the ſurface, it would never return; let $r =$ the earth's radius $= 20961600$ feet, $l =$ the ſpace deſcribed from reſt by bodies which deſcend for one ſecond by the earth's gravity, this * velocity $= \sqrt{4lr} = \sqrt{4 \times 16\frac{1}{12} \times 20961600} = 36722$ feet, or 6.95 miles, that is, the velocity of projection muſt be that of near 7 miles in a ſecond.

* Sect. III. Prop. V. Cor 3.

Cor. 7. If $n = -3$, the velocity of projection above deſcribed is ſuch as is acquired by a body deſcending from reſt through the ſpace $-\frac{r}{-3+1} = \frac{r}{2}$, or half of the radius by the conſtant acceleration of the force 1, which is well known from other principles: for if the centripetal force, which retains a body revolving in a circle, varies in an inverſe triplicate ratio of the diſtance, and a body be projected perpendicularly from the circumference with the ſame velocity as that with which it revolves in the circle, it will never return.

XI.

Let AB repreſent a line void of gravity, Fig. XVI. one extremity of which is fixed at A; let this line be ſuſpended over a wheel B, and kept in tenſion by the weight P. Let W be a ſmall body affixed to C the middle point of the ſtring, and let W be drawn out of its quieſcent poſition through a ſmall ſpace CR, by a force acting in the direction CI, which is perpendicular to AB, as ſoon as this force ceaſes to act, the weight W will be accelerated toward C: having given the length of the ſtring AB, the weights P and W; it is required to determine the time wherein the ſpace RC is deſcribed by W.

Let $RC = a$ be the diſtance from which w begins its motion toward C; then to find the force whereby the body w is impelled towards C, it is to be obſerved, that this force is equal to that which acting in a direction CR would exactly counteract the impelling force ariſing from the tenſion of P. The three forces which ſuſtain the body w in equilibrio, will be in the direction of the three lines RA, RB and CI, which muſt be in the proportion of the three ſides of a triangle, which are reſpectively parallel to thoſe lines: the quantity of the force in the direction CI will

there-

therefore be eaſily determined by drawing BD parallel to AR, and joining AD; then ſince two of the forces are BR and RA or BD, the third muſt be DR, it follows therefore, that the effect of the ſtring's tenſion to impel the body w in the direction IC, is to the tending force P, as DR to RB, that is, ſince DR is very ſmall compared with CB, as $2\,CR$ to CB; in the ſame manner, when w has arrived at any point O, the impelling force at O will be to the weight P, as twice the diſtance OC from the quieſcent point C to half the ſtring's length.

Let the ſtring's length $AB = L$, $CR = a$, $RO = x$, then from what has preceded, the impelling force at $O = \frac{4P \times \overline{a - x}}{L}$, and the force of acceleration at the ſame point $= \frac{4P \times \overline{a - x}}{w \times L}$. Let z be a ſpace through which a body muſt fall from reſt by the action of the earth's gravity to acquire the velocity in O, this will give us $\dot{z} = \frac{4P}{w} \times \frac{\overline{ax - x\dot{x}}}{L}$, and $z = \frac{4P}{w} \times \frac{\overline{2ax - xx}}{2L}$; and if $l = 193$, the velocity at *$O = \sqrt{\frac{8Pl}{wL}} \times \sqrt{2ax - xx}$, let therefore T be put to repreſent the time wherein the body deſcribes PO, we † ſhall have $\dot{T} = \sqrt{\frac{wL}{8Pl}} \times \frac{\dot{x}}{\sqrt{2ax - xx}}$, and the time $T = \sqrt{\frac{wL}{8Pla^2}} \times$ into an arc the verſed ſine of which $= x$, and radius $= a$, and when $x = a$, or when w arrives at C, the time will become $\sqrt{\frac{wL}{8Pla^2}} \times \frac{pa}{2}$, p denoting the number 3.14159: and time of one entire vibration from R to $D = \sqrt{\frac{wL}{2Pl}} \times \frac{p}{2}$ parts of a ſecond, and the number of vibrations in one ſecond $= \sqrt{\frac{2Pl}{wL}} \times \frac{2}{p}$.

An elaſtic ſtring performs its vibrations in a manner not unlike thoſe of the body w, and the time of the ſtring's vibration would be equal to that of w, provided the

* Sect. III. p. 28.

† Sect. III. Prop. III.

the weight of the whole ſtring were collected into the middle point, every thing elſe being the ſame; but according to the real manner wherein an elaſtic ſtring vibrates, it is manifeſt, that the time of vibration will be leſs than that aſſigned in the problem, every thing elſe being given. For according to the preceding ſolution, the whole weight moves over a ſpace *R D*, greater than the ſpace deſcribed by any other point of the ſtring in the ſame time; whereas in the elaſtic vibration, thoſe parts which are nearer to each extremity by vibrating through leſs ſpaces, will deſcribe them in leſs time than in the other caſe, the accelerating force being the ſame.

The difference, however, affecting the vibration of different ſtrings proportionally, the ‡ ratio of the times wherein ſtrings of different weights, tenſions, diameters, and lengths vibrate, may be obtained from the preceding ſolution, as it will appear when theſe deductions are made from the determination of the real time wherein an elaſtic ſtring vibrates. But it may be uſeful to give an example of the time of vibration of a ſtring, deduced from the preceding ſolution, in which it is ſuppoſed, that the whole weight of the ſtring is collected into the middle point, in order that the difference or aberration from the truth ariſing from this aſſumption may be more obvious: ſuppoſe the weight of an elaſtic ſtring were 40 grains, the length 30 inches, the tending force 10000 grains: according to the rule we ſhall have the number of vibrations in one ſecond $= \sqrt{\frac{2Pl}{wL}} \times \frac{2}{p} = \sqrt{\frac{2 \times 10000 \times 193}{40 \times 30}} \times \frac{2}{3.14159} = 36$, and the time of one vibration $= \frac{1}{36}$ part of a ſecond: the true number of vibrations in a ſecond, deduced from the properties of the harmonic curve $= 56.8$ in a ſecond, and the time of one vibration $\frac{1}{56.8}$ part of a ſecond.

‡Helſham's Lectures, p. 273.

This propoſition is here inſerted, chiefly becauſe in ſome ſolutions, according to the hypotheſis above deſcribed, the force which impels the body *w* at any point *R*, has been made ‡ $= \frac{CR}{CB} \times P$, inſtead of $\frac{2\,CR}{CB} \times P$ its true value.

|| Ibid. p. 271.

If the ſtring be imagined always coincident with two right lines *BR*, *AR*, ſo that each may vibrate backward and forward in the manner of a cylindrical pendulum, on axes paſſing through *A* and *B*: this ſuppoſition, it is manifeſt, will not be extremely different from the truth; but in this caſe it will give the time too ſmall on account of the elaſtic ſtring by its curvature extending itſelf further from the axis toward thoſe parts which are adjacent to the middle vibrating point *C*. The times of vibration of the ſame ſtring derived from this hypotheſis, and from that of the harmonic curve, are in the given ratio of 3.14159 to $2 \times \sqrt{3}$, or as 31 to 34 nearly.

The true time wherein an elaſtic ſtring vibrates, is derived from the nature of the harmonic curve, a few properties of which being premiſed, the time of vibration may be deduced from the principles of acceleration, which have been demonſtrated: theſe properties are as follow.

* Smith's Harmonics, p. 248.

1. The * force by which any ſmall particle of an elaſtic ſtring is impelled toward the centre of its curvature, is to the tending force as the length of the particle to the radius of the curvature.

† Ibid. p. 254.

2. The perpendicular † diſtance *DH* of a particle from the ſtring in its quieſcent poſition is to the ſtring's length, as the ſame length is to the radius of curvature multiplied into the number p^2, p being equal to 3.14159, &c. If therefore any perpendicular diſtance *RC* be put $= a$, and the ſtring's length $= L$, we ſhall have the radius of curvature at $R = \frac{LL}{ap^2}$.

3. If the whole ſtring be divided into particles *Bb*, *Dd*, &c. each particle will arrive at the axis *ACB* at the ſame inſtant of time.

It is alſo ſuppoſed, that the greateſt diſtance from the axis *ACB* to which any particle can vibrate, bears no ſenſible proportion to the ſtring's length.

XII.

Fig. XVIII.

Having given the length and weight of an elaſtic ſtring, and the weight which ſtretches it; let it be required to aſcertain how many times the ſtring when put into motion, will vibrate in a ſecond, one vibration being the time elapſed between the ſtring's leaving any given point R, or one ſide of the axis, and the inſtant of its arrival at the ſame diſtance on the other ſide.

Let L be the length of the ſtring, w its weight, P the weight by which it is ſtretched; then will the * ſtring during any inſtant of its vibration coincide with the harmonic curve ADC, then ſince the ſtring's weight $= w$, the weight of the particle $Dd = \frac{Dd}{L} \times w$, and ‡ becauſe the impelling force which acts on Dd is to the weight P, as Dd to the radius of curvature, that is, as § Dd to $\frac{L^2}{p^2 a}$, we ſhall have the force which impels $Dd = \frac{Dd \times p^2 a}{L^2} \times P$, and the weight of Dd being $= \frac{Dd}{L} \times w$, the force which ‖ accelerates the particle Dd at the firſt inſtant of its motion $= \frac{Pp^2 a}{Lw}$. Suppoſe Dd to have deſcribed the ſpace DO, and let $DO = x$, then will $CO = a - x$, and the force which accelerates the particle Dd when at O, will by the ſame method of reaſoning be $= \frac{Pp^2\,\overline{a-x}}{Lw}$: let z be the ſpace which a body muſt deſcribe from reſt by the acceleration of gravity, ſo as to acquire the velocity of the particle Dd in O, this will give $\dot{z} = \frac{Pp^2}{Lw} \times \overline{a\dot{x} - x\dot{x}}$, and $z = \frac{Pp^2}{Lw} \times \frac{\overline{2ax - xx}}{2}$, and the † velocity in $O = \sqrt{\frac{2lPp^2}{Lw}} \times \sqrt{2xa - xx}$.

If T be put equal to the time of deſcribing ⸸ DO, we have

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* Smith's Harmonics, p. 255.

‡ Ibid. p. 248.

§ Ibid. p. 254. Supra p. 94.

‖ Sect. I. Prop. IX.

† Sect. III. p. 28.

⸸ Sect. III. Prop. III.

T

$\dot{T} = \sqrt{\frac{Lw}{2lPp^2}} \times \frac{\dot{x}}{\sqrt{2ax - xx}}$, and taking the fluents $T = \sqrt{\frac{Lw}{2lPp^2a^2}} \times$ into a circular arc, the verſed ſine of which $= x$, and radius $= a$, and when $DO = DC$, that is, when $x = a$, the time $= \sqrt{\frac{Lw}{2lPp^2a^2}} \times \frac{pa}{2}$ $= \sqrt{\frac{Lw}{2lP}} \times \frac{1}{2}$ being the time of deſcribing DC, and the time of one entire vibration $= \sqrt{\frac{Lw}{2lP}}$ parts of a ſecond, and the number of vibrations in one ſecond $= \sqrt{\frac{2lP}{Lw}}$.

Cor. 1. Since the quantity a, or the greateſt diſtance of any part of the ſtring from the axis during one vibration enters not into the expreſſion for the time, it follows, that in whatever ratio the diſtances from the axis to which the ſame ſtring vibrates, the times of vibration will be the ſame, provided thoſe diſtances be very ſmall.

Cor. 2. Let the number of vibrations in a ſecond $= n = \sqrt{\frac{2lP}{Lw}}$, then having given the weight of the ſtring $= w$, and the tending force $= P$, the length of the ſtring which makes n vibrations in a ſecond will be $L = \frac{2lP}{n^2w}$.

Cor. 3. If half the length of the ſtring be diminiſhed in the proportion of the tending force to the ſtring's weight, the time of one vibration will be equal to that wherein a body deſcribes from reſt this diminiſhed length by the force of gravity.

‡ Vid. ſupra p. 95.

Since the velocity ‡ in $O = \sqrt{\frac{2lPp^2}{Lw}} \times \sqrt{2ax - xx}$, when $x = a$, or when D has arrived at C, the velocity will become $= \sqrt{\frac{2lPp^2}{Lw}} \times a$.

Cor. 4. It follows, therefore, that the greateſt velocities generated in a given point of the ſame ſtring, ſtretched by a given weight, but impelled to different diſtances from the axis, are proportional to thoſe diſtances.

To eſtimate the velocity in a particular caſe, let the length of a ſtring = 40 inches, its weight 30 grains, the tending force = 10000 grains, then let any given particle in the ſtring vibrate $\frac{1}{30}$ of an inch from its quieſcent poſition, and the velocity acquired by the particle when coincident with the axis $= \sqrt{\frac{2 \times 193 \times 10000}{40 \times 30}} \times \frac{3.14159}{30} = 5.9392$ inches in a ſecond.

Alſo the velocity acquired by any particle, when it coincides with the axis, may be eſtimated by this general rule; multiply the diſtance of the particle from the axis into the number of vibrations in a ſecond, this product increaſed in the proportion of the diameter of a circle to its circumference, will give the ſpace which the particle would deſcribe uniformly in a ſecond with the velocity acquired.

Cor. 5. An elaſtic ſtring, the weight, length, and tenſion of which are as deſcribed in the problem, will vibrate in the ſame time with a pendulum, the length of which is to the ſtring's * length in a ratio of $w : P \times p^2$, p being = the number 3.14159; for the length of ſuch a pendulum $= \frac{Lw}{p^2 P}$: let this $= z$, and ſince the time in which a pendulum of the length z vibrates in a cycloidal arc is ‡ $p \times \sqrt{\frac{z}{2l}}$, ſubſtituting $\frac{Lw}{p^2 P}$ for z, we have the time wherein the pendulum z, or $\frac{Lw}{p^2 P}$ would vibrate $= p \times \sqrt{\frac{Lw}{2p^2 Pl}} = \sqrt{\frac{Lw}{2Pl}}$, which is the time of one vibration of the elaſtic ſtring by the preceding ſolution.

Cor. 6. Let T repreſent the time of one vibration, then $T = \sqrt{\frac{Lw}{2lP}}$, and $T^2 = \frac{Lw}{2lP}$; let D = the diameter

* Smith's Harmonics, p. 259.

‡ Sect. IV. Prop. I.

of the ſtring, and c a number, which being multiplied into D^2L, will expreſs the weight of the ſtring in the ſame dimenſions with thoſe of P, ſo that w ſhall equal D^2Lc, then we ſhall have by ſubſtituting D^2Lc for w, $T = \sqrt{\frac{D^2L^2c}{2lP}}$, that is, $T = \frac{DL\sqrt{c}}{\sqrt{2lP}}$: and ſince $\frac{\sqrt{c}}{\sqrt{2l}}$ is a conſtant quantity, while the ſpecific gravity of the ſtring is conſtant, it follows, that T will be proportional to $\frac{DL}{\sqrt{P}}$, that is, the time of a ſingle vibration of different ſtrings ſtretched by different weights will be in a ratio compounded of the joint direct ratio of the diameters and lengths, and an inverſe ſubduplicate ratio of the tending forces.

Cor. 7. The tending forces and diameters being the ſame, the time of vibration will be as the ſtring's length; but if the ſame ſtring be ſtretched by different weights, the tending forces will be in an inverſe duplicate ratio of the times wherein the ſtring vibrates.

Cor. 8. Since the times wherein ſtrings vibrate are in a ſubduplicate ratio of their weights and lengths directly, and an inverſe ſubduplicate ratio of the tending forces, the particular note or tone of a given ſtring ſtretched with a given weight, may be known a priori; provided a ſingle experiment be previouſly made, by obſerving the note which is ſounded by a ſtring in given circumſtances. — An † experiment of this ſort was made by the late Dr. Smith, maſter of Trinity College, and is fully deſcribed in his excellent treatiſe of Harmonics.

† Smith's Harmonics, p. 202.

Having fixed a harpſichord wire to a ſmall cylinder of wood, he ſuſpended it ſo as to hang vertically at the ſide of the organ in Trinity Chapel, and by turning round the cylinder, to which the ſtring was affixed, was enabled to regulate its length, ſo that the tone ſhould preciſely coincide with any propoſed note of the organ: the lower extremity of the ſtring was defined by a loop to which a weight of 7 pounds avoirdupoiſe, or 49000 grains, was affixed, and it did not appear that any other terminations were neceſſary, either in the upper or lower extremity, the vibration of the ſtring not being extended lower than the beginning of this loop. When the ſtring's tone became exactly coincident with that of the lower D of the organ, it was found by menſuration, that the ſtring's length was equal to 35.55 inches, and the ſtring being cut at the two terminations, that is, at the beginning of the loop and at the tangent to the wooden cylinder, the vibrating part of it weighed 31 grains. Applying Dr. Smith's or the preceding ſolution to theſe data for the determination of the time in which the ſtring vibrated, we have from the experiment, $L = 35.55$, $P = 49000$, $w = 31$, $l = 193$, and the number of vibrations in one ſecond, was

$$\sqrt{\frac{2 \times Pl}{Lw}} = \sqrt{\frac{2 \times 49000 \times 193}{31 \times 35.55}} = 131 \text{ : that is, the}$$

ſtring in its vibratory motion paſſed the axis AB 131 times in a ſecond. It follows alſo, that ſince the note ſounded on the organ cauſed ſynchronous vibrations in the air, † and becauſe any number of theſe vibrations impreſſes on the ear half that number of impulſes in the ſame time, if by any means the air can be put into a tremulous motion, ſo that 65.5 impulſes ſucceeding each other at equal intervals of time ſhall be impreſſed on the ear in a ſecond, the idea of the ſame note with that deſcribed in the experiment will be excited; for the tone of a note depends not on the loudneſs or ſoftneſs of the ſound, but upon the number of vibrations excited in the air in a given time. Thus a large bell being ſtruck will create the ſame tone as to gravity or acuteneſs, as a flute or other ſimilar inſtrument.

† Newt. Princip. Vol. II. Prop. XLII.

It was inferred from this experiment, that the middle note in the organ denominated D vibrated 4 × 131 = 524 times in a ſecond, and conſequently impreſſed on the ear 262 impulſes in the ſame time.

It may perhaps be uſeful to inſert a few experiments on harmonic ſtrings, whereby the number of vibrations in a ſecond, excited in the air by any given note according to the pitch in preſent uſe may be determined.

The experiment juſt deſcribed would have been ſufficient for this purpoſe, if it were certainly known what note or tone, according to the preſent pitch, is the ſame with the lower D of the organ at Trinity College at the time when Dr. Smith tried his experiment: for at preſent the pitch of this inſtrument is conſiderably different from that which is in common uſe.

In order to aſcertain this point, an inſtrument was made uſe of purpoſely conſtructed by Mr. Ramſden, in the year 1768: the advantages which it poſſeſſes over monochords of the uſual conſtruction are many. The ſtring AB hangs vertically, and its length is terminated at A by an hori-

Fig. XIX.

zontal edge; alſo the other point of termination which in the common monochords, as well as in muſical inſtruments, is a bridge over which the ſtring is ſtretched, is in this conſtruction effected by two ſteel edges *D C* vertically placed; theſe being fixed on a ſrame can be eaſily moved in a vertical direction, ſo as to alter the length of the ſtring in any deſired proportion: theſe edges are ſeparated occaſionally by a ſpring, in order to let the ſtring freely paſs through when its length is altered, and are cloſed again ſo as to preſs the ſtring ſlightly when its length is properly adjuſted. By means of this conſtruction the alteration of the tending force by the application of bridges, &c. is wholly avoided: moreover, that this monochord may be univerſal, that is, may ſerve to exhibit a ſeries of harmonic tones according to any given temperament, viz. Smith's, Huygens', Ptolemy's, &c. the whole length, by means of a ſcale placed immediately under the ſtring, is divided into 100 equal parts, and each of theſe by a micrometer ſcrew *E* ſubdivided into 1000 equal parts, ſo that the length of a given portion of the ſtring may be adjuſted on the monochord true to the $\frac{1}{100{,}000}$ part of the whole length.

Two or three experiments made on this inſtrument may be ſelected for the preſent purpoſe of determining the number of vibrations excited in the air by a given note.

A braſs wire was ſuſpended on the monochord, and being ſtretched by 102.1 ounces weight troy, was ſuffered to remain in that poſition for ſeveral days, in order that the ſtring might not be ſubject to any variation during the experiment: the ſteel edges before deſcribed were adjuſted until the tone of the ſtring was coincident with that of a ſteel key or fork commonly uſed for tuning harpſichords, &c. which correſponded with the loweſt *D* in the baſs. The length of the ſtring appeared by reading it off from the ſcale $= 38.975$ inches, and this length being cut off accurately at the points of termination, the weight of it was found to be 24.35 grains, the tending force was 102.1 ounces, or 49008 grains, we have therefore to obtain the number of vibrations in a ſecond, $P = 49008$, $L = 38.975$, $l = 193$, $w = 24.35$, and the number of vibrations in a ſecond $= \sqrt{\frac{2\,P\,l}{w\,L}} = 141.18$.

In another experiment the length of the ſtring was 40.154 inches, the tending force was 48024 grains, the ſtring's weight $= 22.7$ grains; this ſtring when ſounded was alſo coincident with the ſame note as the former, viz. the lower *D* in the baſs, and by the rule we have the number of vibrations in a ſecond $= \sqrt{\frac{2Pl}{wL}} = 142.6$: the mean of theſe, and of a great many others which it is needleſs to inſert, is 141.89 nearly, which we may eſtimate at 142 without ſenſible error; it follows then, that the tone *D* in the middle of the ſcale excite $142 \times 4 = 568$ vibrations in a ſecond.

It may be added, that out of eight promiſcuous experiments ſimilar to thoſe juſt mentioned, in which the tending forces, lengths, and weights of the ſtrings were in various proportions, the number of vibrations in a ſecond, excited by the ſtring, when in uniſon with the ſame tone, viz. the lower *D*, never varied ſo much as 1.5 vibrations from the mean.

This determination coincides with that of Dr. Smith; for the pitch of the organ of Trinity College is by eſtimation about a hemitone lower than that which is in common uſe: the exact difference is eaſily to be inferred from the experiments above deſcribed: for according to theſe experiments, the vibrations in a ſecond were * 131 and 142 reſpectively, and the difference of theſe tones will be expreſſed by a ſtring divided in the proportion of 131 : 142, which ratio is equal to the ſum of the ratios of 15 : 16, and 98403 : 100000; † that is, ſince 15 : 16 is a hemitone, and the ratio of 80 : 81, denominated a comma, is ſomething leſs than the ratio of 98403 : 100000; it appears, that the pitch of the organ above mentioned is a hemitone, and more than a comma lower than the pitch now in uſe.

This ſolution will enable us to eſtimate the number of vibrations excited in the air by any given note, without difficulty: for example, the loweſt *G* in the ſcale will vibrate once, while the middle *D* ‡ makes $\frac{2}{1} \times \frac{2}{1} \times \frac{3}{2} = 6$ vibrations; therefore, ſince *D* by the preceding determination cauſes 568 vibrations in a ſecond, the lower *G* will excite $\frac{568}{6} = 94\cdot\frac{2}{3}$ vibrations in a ſecond. In the ſame manner it may be inferred, that the higheſt *G* in the ſcale cauſes $568 \times \frac{2}{1} \times \frac{2}{1} \times \frac{4}{3} = 3029$ vibrations in a ſecond. If we add an octave to each extremity of the ſcale, we ſhall pro-

* Supra p. 99.

† Smith's Harmonics, Sect. II.

‡ Ibid. Sect. II.

probably limit the ſenſe of hearing as to the diſtinct perception of harmonic ſounds: this being granted, it will follow, that if the air impreſſes on the ear impulſes however diſtinct, yet if they ſucceed each other with a velocity greater than that of $\frac{3029 \times 2}{2} = 3029$ in a ſecond, or ſlower than at the rate of $\frac{94\frac{2}{3}}{2 \times 2} = 23\frac{2}{3}$ in a ſecond, no diſtinct idea of ſound, as to gravity or acuteneſs, will be conveyed to the mind.

The properties of the elaſtic ſtring inferred from experiment have induced ſome authors, not converſant in the mathematical part of harmonics, to imagine certain analogies between the principles of harmony and of geometrical figures, and to ſuppoſe the properties of each deducible from the other. The celebrated † Tartini fell into ſome fancies of this ſort, which diſcover a conſiderable degree of a ſcientific enthuſiaſm; he found by taking the verſed ſines in a given circle, in arithmetical progreſſion, if the lengths of elaſtic ſtrings (the diameters being the ſame) were as the chords of the arcs correſponding to the verſed ſines above deſcribed, and the tending forces were as the verſed ſines themſelves reſpectively, that all the ſtrings would give the ſame tone, and conſequently vibrate in equal times: in conſequence of this and a great variety of ſimilar deductions of harmonic properties from thoſe of the circle, he concludes, that there is a natural affinity between them, and ſeems to think it conſiſtent that the ſcience of harmony, which he ſuppoſes to be in itſelf perfect, ſhould be immediately related to the circle as the moſt perfect figure.

This analogy, however, obtains only from the known property of the circle, in which the verſed ſines of any arcs are in a duplicate ratio of the correſponding chords, the diameter being conſtant, and in the vibration of elaſtic ſtrings, if the time of vibration and the ſtring's diameter be given, while the length and tending force vary, the tending forces muſt be in a duplicate ratio of the lengths; and a ſimilar analogy might have been as eaſily deduced from the relation between the ordinates and abſciſſas of a parabola, as from that of the verſed ſines and chords of a circle.

Mr. Holder likewiſe, a gentleman of Oxford, in a letter to Sir I. Newton * mentions his having deduced all the harmonic ratios from the 47th prop. of the firſt book of Euclid's Elements, and infers analogies between the properties of harmony not unlike thoſe of Tartini. It is however plain, that theſe ratios may be as well deduced from other figures, as from a right angled triangle; there being few geometrical figures, but what admit of ſimilar deductions of the harmonic ratios. As to the eſſential difference between muſical intervals, called concords and diſcords, there have been various opinions. The following account has been given by ſome authors: it is ſaid that two notes conſtituting a concord, vibrate in times, the ratio of which is expreſſed by whole numbers, either equal or prime to each other and the neareſt to unity. Thus, two ſtrings in uniſon vibrate in times which are in the ratio of equality, and conſtitute the principal of concords. Two ſtrings vibrating in times which are as two to one, conſtitute the octave, which is the concord next to the uniſon in degree of perfection, and ſo on to the interval expreſſed by vibrations which are in the ratio of 2 : 3, 3 : 4, 4 : 5, &c. It is added, that the mind reſts moſt ſatisfied with the perception of thoſe ideas of ratios which are the moſt preciſe and definite; and although the ratio of $\frac{5}{6}$ may excite in the mind as definite an idea as that of $\frac{1}{2}$, when abſtractedly conſidered, yet two lines or two vibrations, which are in the proportion of 1 : 2, will convey a more exact idea of that ratio, than two other lines or vibrations which are in the ratio of 6 : 5, will convey of the latter proportion.

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† Tartini's Principles of Harmony: Stillingfleet's Tranſlation, Chap. I. & Chap. II. p. 18.

* Nugæ Antiquæ.

XIII.

Let AIKN‡ repreſent a cylindrical tube or canal, whereof the axes of the two arms IA, KN are vertical, and that of IK horizontal: ſuppoſe that a fluid were to occupy a part of this tube, ſo that the ſurfaces of the fluid when quieſcent ſhould coincide with EF and CD; let one of theſe ſur-

Fig. XX.

‡ Newt. Princip. Vol. II. Prop. XLIV.

ſurfaces CD be depreſſed by a weight or other means, ſo as to coincide with QH; being depreſſed through the ſpace DH, the ſurface EF will conſequently be raiſed through an equal ſpace EA: then the weight which kept CD depreſſed at QH being ſuddenly removed, the ſurface at AB in the other arm will deſcend by the acceleration of the ſuperior weight in that arm, and by the velocity generated in the whole maſs, when it has arrived at the former or quieſcent poſition EF will ſtill deſcend, and conſequently elevate with a retarded motion the ſurface QH or DC to an altitude MN, equal to that of AB from whence the deſcent began: it is required to aſſign the time elapſed between the inſtant the ſurface AB began to deſcend, and that at which the other ſurface becomes coincident with MN.

Let the axis of the whole tube or canal *AFIKCM* be denoted by *L*, and the force whereby gravity would accelerate the fluid were it unconfined $=1$; then ſuppoſe *EF* to be elevated to *AB*, and conſequently the ſurface *CD* depreſſed through a ſpace $CQ = AE$, and let $EA = a$; then continuing the horizontal line *QH* to *RS*, ſince the fluid contained in the part of the tube *RIKH* is in exact equilibrio, the force whereby the upper ſurface *BA* endeavours to deſcend and to communicate motion to the whole maſs will be the column *AR*, the altitude of which $= 2a$, and this column will be to the whole weight of the fluid as

$2a : L$, wherefore the force which accelerates the deſcent of ‡$AB = \frac{2EA}{L} = \frac{2a}{L}$; in the ſame manner when the ſurface AB has by deſcending arrived at O, the force of acceleration $= \frac{2EO}{L}$, that is, if AO, the ſpace deſcribed from quieſcence, be denoted by x, we have the force at $O = \frac{2 \times \overline{a - x}}{L}$, wherefore if z be the ſpace through which a body muſt fall by gravity from reſt, ſo as to acquire the velocity in O, it will follow, that $\dot{z} = \frac{2a\dot{x} - 2x\dot{x}}{L}$, and V the velocity at $O = \sqrt{\frac{4l}{L}} \times \sqrt{2ax - xx}$: and if T be put to repreſent the time of deſcribing AO, we have $\dot{T} = \sqrt{\frac{L}{4l}} \times \frac{\dot{x}}{\sqrt{2ax - xx}}$, and $T = \sqrt{\frac{L}{4l \times a^2}} \times$ arc of a circle, the verſed ſine of which $= x$, and radius $= a$; and when $AO = AE$, that is, when $x = a$, $T = \sqrt{\frac{L}{4la^2}} \times \frac{pa}{2}$, p being $= 3.14159$, that is, the time in which AB arrives at $EF = T = \sqrt{\frac{L}{4l}} \times \frac{p}{2}$: and the time elapſed between the beginning of the deſcent of AB, and end of the aſcent of QH to $MN = p \times \sqrt{\frac{L}{4l}}$.

Cor. 1. The time of one entire deſcent of the fluid from AB to RS is equal to the time in which a pendulum, the length of which is half the ſum of the tube's axis, performs one vibration; for the time wherein a pendulum, the length of which $= \frac{L}{2}$ makes one vibration $= p \times$ *$\sqrt{\frac{L}{4l}}$.

Cor. 2. The time elapſed between the inſtant AB begins to deſcend, and the inſtant at which it returns to the ſame poſition $= p \times \sqrt{\frac{L}{l}}$, being the time wherein a pendulum, the length of which is equal in length to $2L$, performs one vibration.

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‡ Sect. I. Prop. IX.

* Sect. IV. Prop. I.

Cor. 3. Since the diſtance $AE = a$, enters not into the expreſſion for the time, it follows, that whatever be the altitude AE above the quieſcent poſition, from which the ſurface AB begins to deſcend, the time of the entire deſcent to R will be the ſame, viz. = to $p \times \sqrt{\frac{L}{4l}}$.

Cor. 4. When the lengths of the tubes or canals are different, the time of one entire deſcent will be in a direct ſubduplicate ratio of their lengths.

Cor. 5. During the deſcent of AB to its original quieſcent poſition EF, it will be continually accelerated, and the velocity will be the greateſt at E; this greateſt velocity will be a direct ratio of the diſtance AE, and an inverſe ſubduplicate ratio of the lengths of the tube's axes, that is, the velocity will be in a ſubduplicate ratio of $\frac{EA^2}{L}$.

Cor. 6. The arms IA, KN, are in the propoſition vertical, and at right angles to an horizontal tube of communication IK, but if inſtead of this conſtruction, a bent tube be made uſe of, no alteration will be occaſioned in the ſolution of the problem, or the concluſions derived from it; provided thoſe parts of the tube, through which the ſurfaces aſcend and deſcend, be parallel to each other and perpendicular to the horizon. For example, let Fig. XXI. ACD repreſent a bent tube, the curvature not extending above MN; if this tube be ſo far filled with mercury that the ſurfaces may reſt on a level at AB and CD, when the arms MA, ND are vertical, and motion be communicated to the mercury by inclining the tube a little to one ſide or the other, in a plane coincident with the axes, and then reſtoring it to its vertical ſituation, the mercury will be ſeen to vibrate backward and forward, performing oſcillations through the longeſt and ſhorteſt ſpaces in the ſame time. If the length of the whole tube, i. e. the length of the ſtraight and curve parts taken together, be L, then the time of one oſcillation, that is, the time wherein one ſurface deſcends and the other aſcends from quieſcence to reſt again, will be $= p \times \sqrt{\frac{L}{4l}}$: for example, if the length of the tube be $= 78.4$ inches, the time of one oſcillation will be * $= 3.14159 \times \sqrt{\frac{78.4}{4 \times 193}} =$ 1 ſecond.

The time by experiment is ſomething more than that which is deduced from the theory, as may be expected from the friction of the mercury againſt the ſides of the tube which the theory takes not into account.

The length of the whole tube or canal remaining, let the vertical arms *IA*, *KN* be diminiſhed, and conſequently the horizontal part *IK* increaſed equally, as in fig. XXII. then will the time of one vibration continue as before, and in different canals, the heights of which are very ſmall when compared with their lengths, the times of one vibration will be nearly in a direct ſubduplicate ratio of the breadths, that is, of the horizontal parts of the canal or the diſtances between the vertical arms.

Fig. XXII.

If the undulatory motion of fluids, the † agitations of which are but ſmall, be analogous to the alternate vibrations of the fluid in the canal above mentioned, and ſuppoſing the altitude of the waves to be very ſmall when compared with their lengths, the time of one undulatory motion, that is, the time elapſed between a particle of the fluid's leaving the ſummit of one wave and after its depreſſion again arriving at the ſame altitude, will be in a direct ſubduplicate proportion of the curvilinear diſtance comprehended between the ſummits of the two waves; that is, ſince the heights are very ſmall when compared with the lengths, the time of an undulation will be in a direct ſubduplicate ratio of the diſtances between the ſummits of the two contiguous waves.

† Newton. Princip. Vol. II. Prop. XLII.

Any number of forces may act in ſuch directions and quantities, as to exactly counterbalance each other; a body urged by theſe forces will continue at reſt; in this caſe it is immaterial what be the quantity of matter acted on, as the equilibrium above deſcribed depends only on the quantity and direction of the impelling forces: when one of theſe forces preponderates over the reſt, motion is produced, the quantity and direction of which is likewiſe to be eſtimated by mechanics, but here the matter moved as well as the moving forces muſt be taken into account. In theory we may imagine weights to be collected into points and ſtill to bear any aſſigned proportion to each other, we may alſo imagine, that the materials whereby motion is communicated in mechanical operations are void of gravity, inertia, and the particles of them to move over each other without friction: concluſions produced from theſe data will be mathematically true in theory, but will not be coincident with matter of fact, unleſs the weights of the materials whereby motion is communicated, conſidering theſe weights both as altering the moving forces and the reſiſtance to motion, and the effects of friction be taken into conſideration.

To eſtimate the whole reſiſtance oppoſed to the impelling force, the weights of the materials uſed in the operation muſt be added to the weight moved, together with a weight equivalent to the reſiſtance ariſing from friction. The weights of materials whereof inſtruments and mechanic engines are compoſed being conſtant in the ſame inſtrument or engine, their effects are eaſily allowed for; but the reſiſtance which ariſes from friction, depending on the velocity of motion, as well as various other circumſtances, is not eaſily reduced to geometrical menſuration: yet by mechanic methods it may be almoſt wholly removed whenever it is thought neceſſary to refer principles of motion to deciſive experimental trials; ſo that in any caſe we may with ſufficient certainty aſcertain the reſiſtance which on all accounts oppoſes the communication of motion by a given impelling force. The moving force itſelf is alſo liable to alteration from various cauſes; when the alteration of the moving force, ariſing from the weight of the materials which are uſed in the communication of motion, is conſtant, it is eaſily allowed for by adding this difference to or ſubtracting it from the moving force originally impreſſed: in other caſes, however, it happens that this alteration of the moving force proceeding from the weights of the materials uſed in communicating motion is variable, and may therefore cauſe a variable increaſe or decreaſe of the acceleration. This is frequently the caſe when motion is communicated by means of lines going over wheels, pullies, &c. theſe lines in philoſophical experiments are extremely thin and flexible, and the weight of them is commonly neglected, which may be done generally without error, as far as regards the quantity of matter moved; but it will appear, that the moving force in experiment may become ſo ſmall that the weights of theſe lines ſhall bear a ſenſible proportion to it.

But in all caſes where any phyſical quantity, whether it be weight, time, velocity, &c. is neglected as inconſiderable, the limits of the error occaſioned by this omiſſion ſhould be defined generally for all magnitudes of the variable quantities which are concerned in the problem. If the error of the reſult occaſioned by omitttng any quantity, be found to bear ſo ſmall a proportion to the whole, that it will be either entirely inſenſible in experiment, or of too ſmall conſequence to merit conſideration, it may be ſtill omitted. One inſtance of this kind may be here inſerted, as it relates to the conſtruction of experiments hereafter deſcribed.

Let *BCD* repreſent a fixed pully, over which a line *FEBCDA* is ſuſpended, two equal weights *P*, *P* being affixed to the extremities; then if a weight *w*, heavier than the ſtring *EF*, be added to the higheſt weight, it will ſet the whole in motion, and the ſtring's weight will manifeſtly cauſe a retardation of this motion till the weights on each ſide are at equal altitudes *HH*, and afterwards will increaſe the acceleration: the next propoſition is intended to eſtimate the effects of the ſtring, both in cauſing a variation in the velocities generated as well as in the times of motion.

XIV.

Let BCD be a fixed pully, the weight of which is collected into the circumference and = q; let two weights P and P, be ſuſpended at the extremities of a line going over the pully, and exactly balancing each other. If a weight W be added to either ſide, the weights P + W will preponderate over the other, and will deſcend by continual acceleration; the moving force being conſtantly = W, and the weight moved = 2 P + q + W, pro- Fig. XXII.

provided the ſtring's weight be too ſmall to have any ſenſible effect; but if the ſtring operate ſenſibly, it is manifeſt, that when the moving force W is higher than the weight P on the oppoſite ſide at A, the weight of the ſtring EF will retard the deſcent; but when W has deſcended ſo as to become lower than the aſcending weight P, the ſtring's weight will cauſe an acceleration of the deſcent: having the data above mentioned, together with the length and weight of the ſtring, and EF = the altitude of W + P above the weight P on the oppoſite ſide, at the very beginning of the motion; it is required to determine the velocity acquired by W during its deſcent through a given ſpace.

Let q be the weight of the wheel or pully collected into the circumference, and let the length of the ſtring $= L$, its weight $= p$; alſo let the difference of the altitudes of the weights $P + w$ and P at the firſt inſtant of motion, that is, let $EF = b$, AO the ſpace deſcribed from reſt $= x$, then we ſhall have the moving force at $O = w + \frac{2px - bp}{L} = \frac{Lw + 2px - bp}{L}$, wherefore if $2P + q + p + w$ be put $= Q$, the *accelerating force at $O = \frac{Lw + 2px - bp}{LQ}$, and if z be the ſpace through which a heavy body muſt fall from reſt by gravity, to acquire the velocity in O, we ſhall † have $\dot{z} = \frac{Lw\dot{x} + 2px\dot{x} - bp\dot{x}}{LQ}$, and $z =$

* Sect. I. Prop. IX.

† Sect. IV. Prop. V.

L

$\frac{Lwx + pxx - bpx}{LQ}$, and the velocity at O or $V =$ $\sqrt{\frac{4l \times \overline{Lwx + px^2 - bpx}}{LQ}}$ inches or feet in a ſecond, according to the denomination in which l, and the quantities x, b and L are taken.

Cor. 1. Since the ſquare of the velocity $= \frac{4lwx}{Q} + \frac{\overline{px^2 - bpx} \times 4l}{LQ}$, $\frac{4lwx}{Q}$ being the ‡ ſquare of the velocity if the line were without weight, it is manifeſt, that the quantity $\frac{\overline{px^2 - bpx} \times 4l}{LQ}$ is the variation of the ſquare of the velocity occaſioned by the ſtring's weight; by which it appears, that whenever $b = x$, $\frac{px^2 - bpx}{LQ}$ vaniſhes; that is, when the deſcending weight w is as much lower than the aſcending weight P at the end of the deſcent, as it was above P at the beginning, the velocity is neither increaſed nor diminiſhed by the ſtring's weight. Moreover, this variation in the ſquare of the velocity admits of a maximum, if x is leſs than b, the quantity $\frac{px^2 - pbx}{LQ}$ being greateſt when $x = \frac{b}{2}$, that is, when the aſcending and deſcending weights are on a level; at which poſitions the variation in the ſquare of the velocity of the deſcending weight, occaſioned by the ſtring's weight, will be the greateſt, decreaſing afterwards until it becomes $= 0$ when $x = b$: when x is greater than b, it is plain that $\frac{px^2 - bpx}{LQ}$ admits not of a maximum, increaſing continually as x increaſes.

‡ Sect. III. Prop. V. Cor. 3.

Cor. 2. The variation in the ſquare of the velocity on account of the ſtring's velocity being $\frac{\overline{px^2 - bpx} \times 4l}{LQ}$, it appears, that as long as x is leſs than b, this variation will be negative; that is, the ſtring's weight will diminiſh the velocity generated, until the deſcending weight has deſcribed a ſpace equal to the difference in altitude of the ſtring's two extremities at the beginning of motion, after which the variation abovementioned becoming poſitive, ſhews that the velocity will afterwards be continually increaſed by the ſtring's weight.

XIV.

Every thing remaining as in the laſt propoſition, let it be required to aſſign the time in which the deſcending body deſcribes the ſpace AO from reſt.

* Sect. IV. Prop. XIII.

‖ Sect. III. Prop. III.

Since the velocity at *$O = \sqrt{\frac{Lwx + px^2 - bpx}{LQ}} \times \sqrt{4l}$, if T be the time of deſcribing ‖AO, $\dot{T} = \sqrt{\frac{LQ}{4l}} \times \frac{\dot{x}}{\sqrt{Lwx + px^2 - bpx}}$, and the time required $T =$ fluent of $\sqrt{\frac{LQ}{4l}} \times \frac{\dot{x}}{\sqrt{Lwx \mp px^2 - bpx}} = \sqrt{\frac{LQ}{lp}} \times \log. \frac{\sqrt{px} + \sqrt{Lw - bp + px}}{\sqrt{Lw - bp}}$, (being the entire fluent becauſe it vaniſhes when $x = 0$) which will be a number expreſſing the time in ſeconds, if l be equal to the ſpace through which the earth's gravity accelerates bodies from reſt in one ſecond.

Cor. 1. In order to facilitate the application of the ſolution, the expreſſion for the time ſhould be reduced to an approximate value, which will be mathematically true when the quantity p vaniſhes in reſpect of w or q; and if it be of finite magnitude, the approximate value will deviate leſs from the truth, the ſmaller proportion p bears to w or q. For brevity, let $q = Lw - pb$, which will give the time

T

$T = \sqrt{\frac{L\mathcal{Q}}{lp}} \times \log. \frac{\sqrt{px} + \sqrt{q + px}}{\sqrt{q}}$, and ſince px is incomparably leſs than q, we ſhall have $\sqrt{q + px} = \sqrt{q} + \frac{px}{2\sqrt{q}}$, and $\frac{\sqrt{px} + \sqrt{q + px}}{\sqrt{q}} = \frac{\sqrt{4pqx} + 2q + px}{2q}$, and $T = \sqrt{\frac{L\mathcal{Q}}{lp}} \times \log. \frac{\sqrt{4pqx} + 2q + px}{2q}$; and becauſe $\sqrt{4pqx} + px$ is incomparably ſmaller $2q$, we have $T = \sqrt{\frac{L\mathcal{Q}}{lp}} \times \frac{\sqrt{4pqx} + px}{2q}$, and ſince px vaniſhes in reſpect of $\sqrt{4pqx}$, T will become $= \sqrt{\frac{L\mathcal{Q}}{lp}} \times \frac{\sqrt{4pqx}}{2q} = \sqrt{\frac{L\mathcal{Q}x}{l}} \times \frac{1}{\sqrt{Lw - bp}} = \sqrt{\frac{L\mathcal{Q}x}{l}} \times \frac{2\sqrt{wL}}{2wL - bp}$, (when bp is very ſmall in reſpect of Lw,) that is, T will approximate to $\sqrt{\frac{\mathcal{Q}x}{lw}} \times \frac{2Lw}{2Lw - bp} = \sqrt{\frac{\mathcal{Q}x}{lw}} \times 1 + \frac{bp}{2Lw}$.

Cor. 2. The time wherein the weight would deſcend, if the ſtring were without gravity, would be * $\sqrt{\frac{\mathcal{Q}x}{lw}}$ ſeconds, becauſe $\frac{w}{\mathcal{Q}}$ is the conſtant accelerating force; wherefore the ſtring will increaſe the time (b being greater than x) in the proportion of $2Lw - bp$ to $2Lw$.

* Sect. III. Prop. IV.

Cor. 3. In the preceding corollaries it has been ſuppoſed that p or the ſtring's weight is very ſmall, when compared with the moving force w, but when $\frac{p}{w}$ becomes conſiderable in reſpect of unity, this approximation will deviate from the truth; in this caſe recourſe muſt be had to the expreſſion for the true time derived from the ſolution.

SECT. V.

CONCERNING THE RECTILINEAR MOTION OF BODIES IN FLUIDS.

IN conſidering the properties of accelerated motion, no other reſiſtance has hitherto been ſuppoſed to act in oppoſition to the moving force, except that which ariſes from the inertia of the weights moved; and this inertia always reſiſts the communication of motion with forces proportional to thoſe weights.

In the preceding propoſitions, therefore, relating to acceleration, bodies have been imagined to move in free and unreſiſting ſpace, and the general properties of retarded motions have been deduced from thoſe of acceleration; the forces having been aſſumed in each caſe either as conſtant, or varying according to ſome ratio of the diſtances from a fixed point.

But there are numberleſs forces, both imaginable in theory as well as really operating in the production of natural phenomena, which vary in no proportion of this kind, depending on circumſtances altogether different from thoſe above deſcribed: ſuch, for inſtance, are the forces which vary in ſome ratio of the velocity which may be either direct or inverſe, and may either accelerate or retard the motion of bodies; all theſe caſes being phyſically poſſible, and conſequently admitting of ſolutions derived from the laws of motion.

In the operations of nature, the forces varying with the velocity are chiefly thoſe which are oppoſed by fluids to bodies moving in them. Let a ſolid body formed by the revolution of a plane figure about an axis, be projected or any how impelled in a fluid in the direction of that axis; if there be no other force to interfere with thoſe of impulſe and reſiſtance, the body will continue to proceed in a ſtraight line; and the enſuing propoſitions are intended to inveſtigate from the neceſſary data the ſpaces deſcribed, times of deſcription, and velocities at the end of the given ſpaces or times.

The properties of reſiſting forces which are oppoſed by fluids to bodies which move in them may probably be more obvious, if the nature of fluid ſubſtances as diſtinguiſhed from ſolids be firſt conſidered. Suppoſing all bodies to be formed of elementary hard and ſolid particles of different forms, it appears evident from various experiments, that the force whereby the parts of bodies cohere, depends upon the quantity of ſurface wherein the elementary particles touch each other, as a proximate or immediate cauſe, ſo that if theſe particles be ſpherical, the quantity of ſurface in contact being incomparably leſs than if the particles were cubes, priſms, pyramids, or other ſolid figures terminated by plane ſurfaces, it follows, that the force whereby the particles cohere is in a phyſical ſenſe evaneſcent, which will be the more apparent by the following argument.

Let two ſpheres of equal diameters and ſenſible magnitudes touch each other: the contact will be incomparably leſs than that of any two plane ſurfaces however ſmall; and it is an eaſy propoſition to demonſtrate, that the points wherein equal ſpheres touch each other, are in a direct ratio of their diameters; wherefore, if two ſpheres phyſically evaneſcent be in contact, the quantity of touching ſurface muſt be incomparably leſs, than that between the two ſpheres of finite magnitude which were before mentioned; from all which conſiderations, it plainly appears, upon the hypotheſis of the ſpherical figure of the particles which compoſe fluids, how great muſt be the facility of motion among their parts.

To the preceding arguments add that which follows: if the number of points wherein a given ſpherical particle was touched, increaſed in the ſame proportion as the diameters were diminiſhed, it is evident that the quantity of contact on the ſurface of each ſphere would not be at all diminiſhed from the argument of the ſpheres' diameters being evaneſcent; but ſince when a number of equal ſpheres are included in a ſolid ſpace, no ſingle ſphere can be touched in more than twelve points, it follows, that while the ſpheres' diameters are diminiſhed, the number of contacts in the ſurface of each ſphere remains the ſame, and therefore the quantity of contact exiſting in the ſurface of any one ſphere is ſtill in proportion to the diameters.

Conſequently, if theſe particles be perfectly hard, round and ſmooth, and of evaneſcent magnitude, there will be no reſiſtance to the motion of bodies which im-

impinge on or move through them, except that which ariſes from the inertia of the particles diſplaced, and this conveys to us the idea of a perfect fluid. Few of theſe, however, exiſt in nature, perhaps none whoſe parts are wholly free from friction, coheſion, and tenacity. Air,* mercury, and water, have been eſteemed, as to philoſophical purpoſes, perfect fluids, the coheſion, friction, &c. of their parts, being ſcarcely if at all ſenſible in experiment. In other caſes the degrees of fluidity are various. Of the perfect fluids above mentioned, water and mercury being mixed with heterogeneous ſubſtances, become (with very few exceptions) leſs fluid than before: Of this kind are ſolutions of gums and ſalts, together with oils, balſams, &c. which are, however of a more fluid nature than honey, clays, bitumen, and ſimilar ſubſtances, the degrees of tenacity and coheſion between the particles of which, rank them among ſolid as well as fluid bodies, the properties of both which they ſeem to participate in an imperfect degree. The reſiſtance which ſubſtances of this kind, i. e. imperfect fluids, oppoſed to bodies impinging on them, or moving through them, de-

* Newt. Princip. Vol. II. Prop. XL. ad finem.

depends on the coheſion, tenacity, and friction, as well as the inertia of the matter moved; the geometrical eſtimation of which circumſtances being of no material uſe in phyſical enquiries,[*] the illuſtrious author of this theory, has chiefly conſidered the properties of retardation which bodies ſuffer when moving through the perfect fluids, the coheſion and friction among the parts of which are in a phyſical ſenſe evaneſcent.

Since therefore the reſiſtance which is oppoſed to ſolid bodies, moving in the perfect fluids, proceeds from their inertia only, it is to be next obſerved, that a body when projected or [†] impelled in empty ſpace, will continue its motion uniformly in a ſtraight line; but a body being projected or any how impelled in a fluid cannot proceed in the direction of its motion without diſplacing the fluid, and by communicating [‡] motion to it loſes an equal quantity of its own motion: this gives us the idea of a fluid's reſiſtance, the quantity of which will manifeſtly depend on the form and magnitude of the moving body, and the velocity of its motion: for a greater body, will diſplace a greater quantity of the

[*] Newt. Princip. Vol. II. Prop. XL. ad finem.

[†] I. Law of Motion.

[‡] III. Law of Motion.

fluid than a ſmaller one, every thing elſe being the ſame; and the greater the velocity wherewith a body moves in a fluid, the more motion will be communicated to the fluid (the exact law not being here conſidered) and conſequently loſt to itſelf.

Moreover, the reſiſtance will depend on the fluid's denſity, every thing elſe being the ſame; for it is manifeſt, that it will require more force to diſplace a given quantity of mercury than the ſame quantity of water, and a quantity of water than an equal quantity of air, if the times be equal wherein theſe effects are produced, and the times are always by the definition of forces, both of acceleration and retardation, aſſumed equal in eſtimating the quantities of motion † generated or deſtroyed.

† II. Law of Motion.

Since the parts whereof fluids are compoſed are hard ſolid bodies, ſubject to the ſame general mechanic laws as other bodies, the ratio of the reſiſtances oppoſed to bodies which move in fluids of given denſities, and with given velocities, may be inferred from the principles of mechanics.

I.

If two plane ſurfaces move in fluids, and in directions perpendicular to the reſpective planes, then will the reſiſtances to their motion be in a direct duplicate ratio of their velocities, the joint ratios of the areas of the planes and denſities of the fluids: that is, if $\frac{M}{m}$ be the ratio of the reſiſtances, $\frac{V}{v}$ that of the velocities, and $\frac{A}{a}$, $\frac{N}{n}$ the ratios of the areas and ſpecific gravities reſpectively, the ratio of the reſiſtances will be defined by this equation $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{A}{a} \times \frac{N}{n}$.

For the motion loſt by either plane, for example by A, in a given time, will be the ſame as the motion which would be communicated by the fluid impinging perpendicularly againſt it when quieſcent with the velocity V; and the ſame reaſoning may be applied to the other plane a: this motion communicated in a given time will be as the number of particles and force of each. The ratio of the number of particles impinging on the planes $= \frac{A}{a} \times \frac{V}{v}$, and the ratio of the forces of each particle reſpectively $= \frac{V}{v} \times \frac{N}{n}$, ſo that the ratio of the quantities of motion communicated to the quieſcent planes in a given time, by the fluids impinging perpendicularly with the velocities V, v, will be the ſum of the ratios $\frac{A}{a} \times \frac{V}{v} \times \frac{V}{v} \times \frac{N}{n}$; that is, the ratio of the quantities of motion loſt in a given time by the planes A, a, moving with the velocities V, v reſpectively, or the reſiſtances oppoſed to their motion will be defined by the equation $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{A}{a} \times \frac{N}{n}$.

Cor. 1. In this propoſition the direction of the plane's motion has been aſſumed always perpendicular to the planes themſelves. Now, let their motion be in a direction any how inclined to the planes, the number of the particles will be diminiſhed in the ratio of the radius to the ſine of the inclination above deſcribed; alſo, the force of each particle will be diminiſhed in the ſame ratio: wherefore on both accounts the reſiſtance determined as before will be diminiſhed in a duplicate ratio of the radius to the ſine of inclination; let therefore $R =$ radius, and S, s the ſines of the angles in which the planes A, a are inclined to the directions of their motion, and it follows, that to eſtimate the ratio of the reſiſtances to the two planes, the ratios $\frac{S^2}{R^2} \times \frac{R^2}{s^2}$ or $\frac{S^2}{s^2}$ muſt be added to the ratio of the reſiſtances before determined, ſo that on all accounts $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{A}{a} \times \frac{N}{n} \times \frac{S^2}{s^2}$.

Cor. 2. If equal planes move in a given fluid, and in directions equally inclined to themſelves, then becauſe in this caſe $\frac{A}{a} = \frac{N}{n} = \frac{S^2}{s^2} = 1$, we have $\frac{M}{m} = \frac{V^2}{v^2}$; that is, the reſiſtances are in a duplicate ratio of the velocities.

Cor. 3. If the ſame body of any ſhape, moving in a given fluid with different velocities, always preſerves the ſame poſition in regard to the direction of its motion, the reſiſtances will be in a duplicate ratio of the velocities.

The preceding propoſition expreſſing the proportion of the reſiſting forces only, will not enable us to aſcertain the abſolute quantity of reſiſtance, which a body actually meets with in paſſing through a medium, but which muſt neceſſarily be known, in order to determine the retardation or decrement of velocity in a given time. As when bodies are accelerated the moving force is eſtimated by weight *, ſo the reſiſting force in retarded motions is alſo equi-

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* Sect. I. Prop. X.

equivalent to a weight: when a body thrown perpendicularly upwards in empty ſpace is reſiſted by the force of gravity, this reſiſtance is always equal to the body's weight, ſo that the retardation of all bodies is the ſame; but the reſiſtance oppoſed to bodies moving in fluids, is a weight which varies with the velocity being in the duplicate ratio thereof, and the eſtimation of this reſiſtance a priori, from the neceſſary data, is attended with conſiderable difficulties, involving the conſideration of impact relatively to weight: for in this caſe, it muſt be determined what weight is equal to the impact of the body moving againſt the particles of the fluid, or which is the ſame, the impact of the particles of fluid ſtriking againſt the ſolid when quieſcent with the ſame velocity.

Sir I. Newton truſted not wholly to theory either for the determination of the ratio of the reſiſtances, nor the abſolute quantity of them; but inſtituted various experiments, wherein it appeared that the reſiſtances were by ſome trials in a ratio rather greater †, and by others in a ratio ſomething leſs than the duplicate ratio of the velocities: theſe differences being no more than what the unavoidable imperfections in experiments of this kind might occaſion, he was at length ſatisfied, that the reſiſtances to a given body moving in the ſame fluid were in a duplicate ratio of the velocities, under certain reſtrictions and conditions, &c. hereafter deſcribed ‖.

† Newton. Princip. Vol. II. Prop. XL. Scholium.

‖ p. 135.

He alſo found that the reſiſtance ‡ oppoſed to a globe, moving uniformly in a fluid, was to the conſtant reſiſting force or weight whereby the whole motion of the globe would be deſtroyed, during the time of its deſcribing $2\frac{2}{3}$ diameters uniformly with the given velocity, as the denſity of the fluid to the denſity of the ſphere, which gives the abſolute quantity of reſiſtance oppoſed to the body's motion, the denſities of the ſphere and fluid, the diameter of the ſphere, and velocity of its motion being known, as will hereafter appear.

‡ Newt. Princip. Vol. II. Prop. XXXVII.

The principles of reſiſtance being diſcovered by Sir I. Newton, and confirmed fully and eſtabliſhed by the ſucceeding age, the ſubject may be probably more obvious if conſidered ſynthetically; which method, though unfit for making diſcoveries, is eſteemed the moſt proper to exhibit the known principles of ſcience in regular and ſyſtematic order. The following propoſition is intended to determine the abſolute reſiſtance oppoſed to a body moving in a fluid with a given velocity, that is, to determine a weight equivalent to that reſiſtance.

II.

A plane ſurface moving in a fluid, in a direction perpendicular to the plane, is oppoſed by a reſiſtance equal to the weight of a column of the fluid, the baſe of which is the reſiſted ſurface, and the altitude equal to that through which a body muſt fall from reſt by the acceleration of gravity to acquire the velocity of the moving plane.

Fig. XXIV. Let the line *AB* repreſent the projection of the plane moving in the direction *CD*, which is perpendicular to it; and ſuppoſe that during its retarded motion it has advanced from its poſition *AB* to *ab*, deſcribing the elementary ſpace or parallelopepid *ABab*, with a given velocity *V*. Then will the plane *AB* communicate motion to the particles diſplaced, and will itſelf loſe an equal quantity; and by the principles of bodies' colliſion, the quantity of motion loſt by the reſiſted plane will be the ſame, as that which the particles would communicate to it in the ſame time, were the plane quieſcent, and the particles impinged upon it with the ſame velocity. We are only then to aſcertain what weight is equivalent to the impact of a given quantity of fluid ſtriking againſt the plane with a given velocity, and communicating motion to it while it is deſcribing an evaneſcent ſpace. Let a cylindrical veſſel, the baſe of which is horizontal, be filled with a fluid equally denſe with that which reſiſts the moving plane above deſcribed, the altitude of the fluid above the baſe being equal to that from which a body muſt fall by the acceleration of gravity, ſo as to acquire the velocity of the plane's motion: then if an ‡ aperture be made in the baſe, the fluid will ruſh forth with a velocity equal to that of the plane's motion.

‡ Bernoulli Hydrodynamica. Sect. IV.

Now, let the plane be applied under the aperture and parallel to it, and the veſſel being kept conſtantly full, let the plane be continually raiſed ſo as to approach the aperture: the fluid will ſtrike againſt it with a continually decreaſing velocity, till at laſt when the plane is juſt contiguous to the aperture, the motion will be evaneſcent; and when the motion of the fluid is vaniſhing, the plane will ſuſtain the weight of the incumbent column, that is, a column of the fluid, the baſe of which is the plane preſſed, and height equal to that of the fluid's ſurface from the aperture: but the velocity of the fluid when juſt iſſuing out of the aperture, is equal to that which a heavy body acquires in falling through the perpendicular diſtance, between the aperture and fluid's ſurface; wherefore the naſcent impulſe of a fluid, acting perpendicularly on the plane, is equal to the weight of a column of the fluid: the baſe of which equals the aperture, the altitude being the ſame with that of the fluid's ſurface above the aperture: but the reſiſtance to the plane from the fluid, is equal to the motion communicated to the plane by the naſcent impulſe of the fluid ſtriking it when quieſcent with a velocity equal to that of the plane's motion; wherefore the reſiſtance to a plane, when moving in a fluid in a direction perpendicular to itſelf, is equal to the weight of a column of the fluid, the baſe of which is the reſiſted ſurface, and altitude equal to that from which a body muſt fall from reſt by the acceleration of gravity to acquire the velocity of the plane's motion.

Cor. 1. The reſiſtances to the ſame body moving with different velocities in the ſame fluid, are in a duplicate ratio of their velocities, becauſe the baſe of the columns of fluid equal to the reſiſtances being given, their altitudes will, by the propoſition, be in a duplicate ratio of the velocities wherewith the plane moves. The other corollaries alſo which were deduced from the firſt propoſition follow from this by the ſame methods of inference.

Cor. 2. If a cylinder moves in a fluid, ſo that the direction of its motion ſhall always coincide with the axis of the cylinder, the anterior plane ſurface only will communicate motion to the fluid, becauſe the curved ſurface being parallel to the direction wherein the whole body moves, neither accelerates nor retards the particles of the fluid; theſe, as well as the cylinder, being ſuppoſed ſo ſmooth that no friction can have ſenſible effect. Let then $d =$ the diameter of the cylinder, $p = 3.14159$, &c. V the velocity of the cylinder's motion, z the altitude from which a heavy body muſt fall to acquire the velocity V; moreover, let the ſpecific gravity of the fluid be expounded by 1; then will $\frac{d^2 p}{4} =$ the area of the cylinder's baſe, and

$\frac{z d^2 p}{4}$ will be the magnitude of a cylinder, whoſe diameter is d, and altitude $= z$, and the weight of a cylindrical column of the fluid, the baſe of which equal the baſe of the cylinder and altitude $= z$, is $\frac{z d^2 p \times 1}{4}$, which is equal to the reſiſtance † oppoſed to a cylinder moving in a fluid, the denſity of which is $= 1$, the cylinder's velocity being $V = \sqrt{z}$, when referred to the general ſtandard 1 ||, or $= \sqrt{4 l z}$, when referred to the velocity of $2 l$ in a ſecond, which gravity generates in one ſecond of time.

† Supra p. 125.

|| Sect. III. Prop. V.

Cor. 3. Let the weight of the cylinder $= w$, and let $F =$ its retardation, meaſured by the velocity deſtroyed in a given time in reference to that which is deſtroyed or generated by gravity in the ſame time; then ſince we have in ‡ general $\frac{F}{f} = \frac{M}{m} \times \frac{q}{Q}$, that is, the ratio of the forces of retardation is compounded of the direct ratio of the reſiſting forces, and the inverſe ratio of the weights moved, aſſuming f, m and q each $= 1$ *, it will be $F = \frac{M}{Q}$, and ſince by Cor. 2. $M = \frac{z d^2 p}{4}$, and $Q = w$, we have $F = \frac{z d^2 p}{4 w}$, the force which retards the cylinder in the circumſtances above deſcribed.

‡ Sect. I. Prop. IX.

* Sect. II. Prop. VII.

The reſiſtance to a plane moving in a fluid with a given velocity, and in a direction always perpendicular to itſelf, may be denominated the whole reſiſtance, to diſtinguiſh it from the diminiſhed reſiſtances, which are oppoſed to the plane when moving with the ſame velocity, but in directions inclined to the plane at various angles; and if the baſe of a column of the fluid be equal to the moving plane, and the altitude the ſame as that from which a body muſt fall by the conſtant acceleration of gravity to acquire the velocity of the plane's motion, the weight of this column diminiſhed in a duplicate proportion of radius to the ſine of the angle at which the plane is inclined to the direction of its motion, will be the reſiſtance to the plane's motion eſtimated in a direction perpendicular to the *plane; but the reſiſtance eſtimated in the direction of the plane's motion will be different from the former as will appear by the ſubſequent propoſition. * Sect. V. Prop. I. Cor. 1.

III.

Let AB repreſent a plane, moving in a fluid in the direction AF, with ſuch a velocity as would be acquired in falling from reſt by the acceleration of gravity through a ſpace = AF. Then will the reſiſtance to the plane, eſtimated in the direction of its motion AF, be equal to the weight of a column of the fluid, whoſe baſe = the plane, and altitude = the ſpace AF diminiſhed in a triplicate ratio of the radius to the ſine of the angle at which the plane is inclined to the direction of its motion. Fig. XXV.

For the reſiſtance eſtimated in the direction perpendicular to the plane is equal to the weight of a column of fluid, the baſe of which is the plane, and altitude *AF*, diminiſhed in the duplicate † proportion of the radius to the ſine of the angle in which the plane is inclined to the direction of its motion. In the line *AF* take *AC* to *AF* as the ſine of inclination, that is, the ſine of *CAB* to radius; through *C* draw *CB* perpendicular to *AB*, then will *AF* be to *BC* in a duplicate ratio of the radius to the ſine of incidence *CAB*: the reſiſtance, therefore, in the direction *BC* will be the weight of a column of the fluid, the baſe of which is the ‡ plane, and altitude *BC*. † Sect. V. Prop. I. ‡ Sect. V. Prop. I.

The line *BC* will therefore repreſent the force of reſiſtance in that direction, the plane's ſurface remaining the ſame: to eſtimate the reſiſtance in the direction *AC* reſolve *BC* into two, whereof *DC* is in the direction *CA* and *BD* perpendicular to it; then the force *DB* will not contribute to retard the progreſs of the plane in the direction AC, but will impel it in a lateral direction only, and CD will be the only force of reſiſtance acting in the direction oppoſite to that of the plane's motion: but ſince $AF : AC$, CB and DC are in a continual proportion, we ſhall have $AF : DC : AF^3 : AC^3$, or as $AC^3 : BC^3$, that is, the whole or greateſt reſiſtance, if it were perpendicular to the plane is to the reſiſtance eſtimated in the direction CA, in a triplicate ratio of radius to the ſine of inclination.

Cor. 1. Let another plane IB be joined to the former at an angle $ABI =$ twice the angle of inclination BAD, and then let BI be taken $= BA$, and draw IK parallel to AC: alſo draw BL perpendicular to IK, and ſuppoſing theſe two planes to move in the direction AC or IL, then will the reſiſtance to the plane IB eſtimated in the direction of its motion, be equal to the weight of a column of fluid, the baſe of which is equal to IB, and altitude the ſpace due to the *velocity of the plane, diminiſhed in a triplicate ratio of the radius to the ſine of inclination, wherefore the whole reſiſtance to both planes in the direction of their motion will be the weight of a column of the fluid, the baſe of which is equal to the planes, and altitude the ſame with that juſt deſcribed. Moreover, the lateral force BD will be counteracted by an equal and oppoſite force LB; and conſequently the remaining forces of reſiſtance being in the direction KL or BC will only retard the progreſs of the planes, but will not alter the direction of their motion.

Fig. XXVI.

[*] Sect. V. Prop. III.

Cor. 2. Let any ſolid formed by the revolution of a plane figure round an axis, be projected or any how impelled in a fluid in the direction of that axis, and let the planes of two circles be drawn perpendicular to the axis, and contiguous to each other: then will the circumferences of the circles include an annular ſpace, which will be the fluxion of the ſurface of the ſolid. If a tangent be drawn to this ſurface in the ſame plane with and meeting the axis, it appears from the preceding corollary, that the reſiſtance to the annulus or elementary ſpace, will be equal to the weight of a column of the fluid, the baſe of which is the annulus, and the altitude equal to the ſpace due to the velocity with which the ſolid moves, diminiſhed in a triplicate ratio of the tangent to the ordinate correſponding.

Cor. 3. The weight deſcribed in the laſt Cor. will be the fluxion of the reſiſtance, if this be expreſſed in terms of the abſciſſa and conſtant quantities, the entire fluent will give the whole reſiſtance oppoſed to the ſolid, eſtimated in the direction of its motion.

IV.

IV.

If a ſphere and cylinder of the ſame diameter, move with equal velocities in the ſame fluid, and if the direction of the cylinder's motion, coincides with its axis, the reſiſtance oppoſed to the motion of the globe, will be to the reſiſtance oppoſed to the cylinder in the ratio of one to two. Fig. XXVII Newt. Princip. Vol. II. Prop. XXXIV.

The fluid in which the ſphere and cylinder move is ſuppoſed to be ſo compreſſed, that the preſſure on every part of the moving bodies ſhall be the ſame, as when they are at reſt: moreover, it is aſſumed as true, that the hinder part of the ſolids contributes nothing to the reſiſtance, which will be the ſame as if the anterior part only were expoſed to the fluid, the weight and every thing elſe remaining.

Let *CT* be the line which the ſphere's centre *C* deſcribes during its motion, the velocity of *C* being equal to that which is acquired by a heavy body which deſcends from reſt by the acceleration of gravity through the ſpace z. Alſo, let *GD* be a diameter perpendicular to *CT*, *GED* being the plane of a ſemicircle, which paſſes through *CT*. In the ſemicircle *GED* draw any radius *CA*; draw *AB* perpendicular to *CE*, *ab* parallel and contiguous to *AB*, and *am* parallel to *CT*; moreover, through the point *A* draw *AT*, a tangent to the circle at *A*.

Suppoſing the plane *CEG* to revolve round *CET* as an axis, the evaneſcent arc *Aa* will generate an evaneſcent ſurface, which will be the fluxion of the ſurface of the ſphere: if this fluxional area, which we may denote by b, moved in the fluid in a direction perpendicular to its plane with the velocity, which a heavy body would acquire by falling freely through a ſpace z, the reſiſtance to its motion would be the weight of a column of the fluid, of which the ‡magnitude $= bz$; but as this evaneſcent ſurface is inclined to the direction of its motion *CT* in an angle *ATC*, the whole reſiſtance bz muſt be diminiſhed in a triplicate ratio of radius to the ſine of *ATC*;

‡ Sect. V. Prop. I.

ATC; that is, in the * ratio of $AT^3 : AB^3$; which will give the fluxion of the reſiſtance to the ſolid in the direction of its motion $CT = \frac{bz \times AB^3}{AT^3}$ whatever be the curve GE; and ſince GE is the quadrant of a circle, and $\frac{AB^3}{AT^3} = \frac{CB^3}{CA^3}$ the fluxion of the reſiſtance $= bz \times \frac{CB^3}{AC^3}$: making $EC = r$, $CB = x$, $Bb = \dot{x}$, $p = 3{\cdot}14159$, &c. from the properties of the circle, $AB = \sqrt{r^2 - xx}$, and by ſimilar triangles $Aa : ma :: AC : AB$, or $Aa : \dot{x} :: r : \sqrt{r^2 - x^2}$, which gives $Aa = \frac{r\dot{x}}{\sqrt{r^2 - x^2}}$: moreover, the circumference of the circle generated by the point A, while GEC revolves round CE as an axis, will be $2p \times \sqrt{r^2 - x^2}$, which being multiplied into Aa or $\frac{r\dot{x}}{\sqrt{r^2 - x^2}}$ will give $2pr\dot{x}$, for the fluxion of the ſurface of the ſolid; and z being the ſpace due to the velocity of the ſphere's motion, $2prz\dot{x}$ will be the whole reſiſtance to the ‡ evaneſcent reſiſted ſurface, ſuppoſing the direction of motion perpendicular to it; this being diminiſhed|| in the ratio of $r^3 : x^3$, or being multiplied into $\frac{x^3}{r^3}$ gives $\frac{2przx^3\dot{x}}{r^3}$, for the reſiſtance to the evaneſcent area, eſtimated in the direction of its motion CT, and the reſiſtance to the whole ſurface = the fluent of $\frac{2przx^3\dot{x}}{r^3} = \frac{pzx^4}{2r^2}$, or when $x = r$, the reſiſtance to the hemiſphere $= \frac{pzr^2}{2}$: but the reſiſtance § to the cylinder moving with the ſame velocity in the direction of its axis is $\frac{pd^2z}{4}$ or pr^2z, which is to the reſiſtance juſt found $\frac{pr^2z}{2}$ as 2 : 1.

If the diameter of the ſphere = d, the ſpace due to its velocity = z: the reſiſtance to the ſphere's motion will = $\frac{pd^2z}{8}$, the ſpecific gravity of the fluid being = 1.

* Sect. V. Prop. III. Cor. 2.

‡ Sect. V. Prop. I.

|| Sect. V. Prop. III. Cor. 2.

§ Sect. V. Prop. II. Cor. 2.

V.

The force of reſiſtance which is oppoſed to a ſphere, moving in a fluid with any given velocity, is to the force which would deſtroy the ſphere's whole motion, in the ſame time in which it deſcribes uniformly $\frac{8}{3}$ parts of its diameter, as the denſity of the fluid to the denſity of the ſphere.

Newt. Princip. Vol. II. Prop. XXXVIII.

Let the ſpecific gravity of the ſphere be to that of the fluid as $n : 1$, alſo let M be the reſiſting or moving force which would deſtroy or generate the ſphere's motion in the time deſcribed in the propoſition: let the ſphere's weight $\frac{d^3 p n}{6} = \mathcal{Q}$, V the velocity of the ſphere's motion,* $z = \frac{V^2}{4 l}$ the altitude, from which a body muſt fall from reſt by the acceleration of gravity to acquire the velocity V: then will a body deſcribe the ſpace ‡ $\frac{8 d}{3}$ uniformly with the velocity V, in the ſame time wherein it deſcribes $\frac{4 d}{3}$, when it is retarded or accelerated by a conſtant force $\frac{M}{\mathcal{Q}}$, which deſtroys or generates its whole velocity V. To find, therefore, the reſiſting force M, or retarding force $\frac{M}{\mathcal{Q}}$, which will deſtroy the whole motion of the ſphere projected with the velocity V, while it deſcribes a ſpace $= \frac{4 d}{3}$ by an uniformly retarded velocity, we ſhall have † $V = \sqrt{\frac{M}{\mathcal{Q}} \times \frac{4d}{3} \times 4 l}$, but ‖ $V = \sqrt{4 l z}$, wherefore $\frac{M}{\mathcal{Q}} \times$

* Sect. III. Prop. V.

‡ Sect. III. Prop. II.

† Sect. III. Prop. V.

‖ Supra.

$4 d$

$\frac{4d}{3} = z$, and $M = \frac{3Qz}{4d}$; or ſince $Q = \frac{pd^3n}{6}$, it follows that $M = \frac{zpd^2n}{8}$, the uniform reſiſting force which would deſtroy the ſphere's whole motion in the ſame time it deſcribes the ſpace $\frac{4d}{3}$ with an uniformly retarded velocity, or $\frac{8d}{3}$ with the firſt velocity of projection continued uniform; which velocity is in both caſes equal to that acquired by a heavy body deſcending from reſt through the ſpace $z = \frac{V^2}{4l}$. But the reſiſting force oppoſed to the ſphere's * motion $= \frac{zpd^2}{8}$, which is to $\frac{zpd^2n}{8}$, the weight or reſiſtance which would generate or deſtroy the ſphere's velocity V, the ſame time in which the ſphere deſcribes uniformly the ſpace $\frac{8d}{3}$, with the velocity of projection V, as 1 to n; that is, as the ſpecific gravity of the fluid to that of the ſphere.

* Sect. V. Prop. IV.

VI.

Fig. XXVIII.

Let a ſphere of given diameter be projected in a fluid, the ſpecific gravity of which is to the ſpecific gravity of the ſphere as 1 : n: having given the velocity of projection from the point c, let it be required to aſſign the velocity of the ſphere's motion at any given point o.

Let the velocity of projection be that which a heavy body acquires in deſcribing the ſpace b from reſt; alſo let the ſphere's diameter $= d$, and $CO = x$, $p = 3.14159$, $l = 193$ inches, z the ſpace through which a body deſcends by gravity from reſt to acquire the ſphere's velocity in O, then the only force which acts on the ſphere while it is deſcribing the evaneſcent ſpace $O\,o$, is that which retards its motion, and is $= -\frac{3z}{4dn}$. If therefore $Oo = \dot{x}$, we ſhall have $\dot{z} = -\frac{3z\dot{x}}{4dn}$, and $\frac{\dot{z}}{z} = -\frac{3\dot{x}}{4nd}$, and taking the fluents log. $z = -\frac{3x}{4nd}$, which ſhould vaniſh together, ; but when $x = 0$, it follows from the problem, that $z = b$, conſequently log. $z =$ log. b; the entire fluents therefore will be log. $z =$ log. $b - \frac{3x}{4nd}$, and $\frac{3x}{4nd}$ $=$ log, $b -$ log. $z =$ log. $\frac{b}{z}$: let $e = 2.71828$, being the number the hyperbolic logarithm of which $= 1$, and we ſhall have $\frac{b}{z} = e^{\frac{3x}{4nd}}$, and $z = b \times e^{-\frac{3x}{4nd}}$, and the ſphere's velocity at O or $V = \sqrt{b \times e^{-\frac{3x}{4nd}}}$ when the velocity is referred to the general ſtandard 1, but if it be compared with the velocity $2\,l$ in a ſecond which gravity† generates in deſcending bodies in one ſecond of time, we ſhall have $V = \sqrt{4\,l\,b \times e^{-\frac{3x}{4nd}}} = \frac{\sqrt{4\,l\,b}}{e^{\frac{3x}{8nd}}}$.

Cor. 1. The velocity loſt by a globe projected in a fluid during the time of its deſcribing the ſpace x, will be equal to that part of the initial velocity which is expreſſed by the fraction $\frac{e^{\frac{3x}{8nd}} - 1}{e^{\frac{3x}{8nd}}}$, the letters e, x, n, d ſignifying as before.

Cor. 2. Thus, ſuppoſe a globe be projected in a fluid of the ſame denſity, and to deſcribe a ſpace equal to three of its diameters, then $x = 3\,d$, and $\frac{3x}{8nd} = \frac{9}{8}$, and the velocity loſt $= \frac{e^{\frac{9}{8}} - 1}{e^{\frac{9}{8}}} = \frac{11.48}{17}$, that is, a globe moving in

† Sect. III. p. 18.

in a reſiſting medium of the ſame denſity with itſelf through a ſpace equal to 3 of its diameters, loſes $\frac{11.48}{17}$ parts of the velocity with which it was projected. It may be here remarked, that the editor of Dr. Helſham's lectures, from the ſame data ‡ makes the loſt velocity $\frac{9}{17}$ inſtead of $\frac{11.48}{17}$ of the whole velocity of projection; but his expreſſions for the velocity loſt by a body moving through a reſiſting medium, are by no means deducible from or conſiſtent with the Newtonian propoſitions which are there referred to.

‡ Helſham's Lectures, p. 366.

Cor. 3. From this ſolution we may obtain the quantity of motion which would be loſt in any given time by the earth, if the medium through which it moved were of equal denſity with the atmoſphere at the earth's ſurface. Suppoſe the time of motion to be one year, and ſince the ſun's parallax is 8″6, the ſun's diſtance from us = about 24000 of the earth's radii, or 12000 diameters, it follows, that in the circumference of the earth's orbit there are contained 24000 × 3.14159 = 75398 diameters, and in one year the earth will have moved through a ſpace equal to 75398 of his diameters, and if the mean denſity of the earth be to that of air || as $5 : \frac{1}{800}$, or as 4000 : 1, we ſhall have in the preceding expreſſion for the velocity loſt $n = 4000$, $\frac{x}{d} = 75398$, and $\frac{3x}{8nd} = 7.068$, then will the velocity loſt $= \frac{e^{\frac{3x}{8nd}} - 1}{e^{\frac{3x}{8nd}}} = \frac{1173}{1174}$ part of the whole.

|| Newt. Princip. Vol. III. Prop. X.

Cor. 4. When the quantity $\frac{3x}{8nd}$ is very ſmall, the velocity loſt will approximate very nearly to $\frac{3x}{8nd}$, becauſe in that caſe $e^{\frac{3x}{8nd}} = 1 + \frac{3x}{8nd}$, and $\frac{e^{\frac{3x}{8nd}} - 1}{e^{\frac{3x}{8nd}}} = \frac{3x}{8nd}$:

thus to aſſume an *example from Sir I. Newton, let a globe of the ſame denſity with water be projected through a medium of the ſame denſity with that of air near the earth's ſurface, and let it be required to aſſign the motion loſt during the time the ſphere is deſcribing a ſpace equal half its diameter; here we have $x = \frac{d}{2}$, $n = 860$, $\frac{3x}{8nd} = \frac{3}{13760} = \frac{1}{4586}$, ſo that the globe will have loſt that part of its whole velocity which is expreſſed by the fraction $\frac{1}{4586}$. This ſhews how nearly this approximation is to the true value; the error not being ſenſible in the expreſſion above deduced, which coincides with that of Sir I. Newton to an unit in the denominator.

* Newt. Princip. Vol. II. Prop. XL. in Scholio.

Cor. 5. It follows alſo, that if the velocity of projection from *C* be to that of the ſphere when at any point of the ſpace deſcribed *O*, as m to 1, then will the ſpace *CO* or $x = \frac{8nd}{3} \times \log. m$. Thus, ‡ ſuppoſe a ſphere moving in a fluid of the ſame denſity with itſelf loſes half its velocity, then will $m = 2$, $n = 1$, and the ſpace deſcribed by the ſphere $= \frac{8d}{3} \times \log. 2 = \frac{8d \times .6931472}{3} = 1.84839 \times d$; the ſphere therefore loſes half its velocity before it has deſcribed a ſpace equal to 2 of its diameters.

Fig. XXVIII.

‡ Newt. Princip. Vol. II. Prop. XXXVIII. Cor. 4.

The theory of reſiſtances, † oppoſed to bodies moving in the perfect fluids, is demonſtrated by Sir I. Newton only under certain conditions and reſtrictions, which may be here mentioned.

† Ibid. Vol. II. Prop. XL. &c.

1. The particles of fluid wherein the body moves are ſuppoſed to be perfectly nonelaſtic.

2. The fluid is imagined to be infinitely compreſſed.

The former condition is ſtrictly applicable to the perfect fluids, mercury, water, and even to air, when the reſiſted body moves ſlowly, in which caſe the aerial particles ſingly conſidered, may be juſtly imagined to ſlide away from bodies moving through them without adheſion, in the ſame manner as thoſe particles which compoſe water or mercury. This is only meant to diſtinguiſh the nonelaſtic particles from thoſe which by rebounding from an impact would tend to increaſe the reſiſtance beyond that which is aſſigned in the theory. The other condition, i. e. that of infinite compreſſion, obtains not in any fluid whatever, ſome allowance therefore is due on this account: but in experiments made on bodies moving very ſlowly, this allowance will be of no material conſequence, becauſe from the ſlowneſs of the motion, the parts left by the ſphere during its progreſs will be inſtantly occupied by the ſurrounding fluid; but when the velocity of projection is conſiderably increaſed, the laws of reſiſtance demonſtrated by Sir I. Newton, depending on the effects of the inertia of the matter diſplaced only, muſt not be expected to correſpond with matter of fact. Thus, cannon or muſket ſhot projected with velocities from 400 to 1600 feet in a ſecond, by their great velocities leave behind them during their paſſage through the air, either partial or entire vacuity; when the velocity is equal to that of 1600 feet in a ſecond, the ſpace deſcribed by the ball as it moves along, is an abſolute vacuum, for the air by its elaſticity ruſhes into empty ſpace with a velocity of no more than 1296 feet in a ſecond, and will not therefore move with ſufficient celerity to fill up the ſpace deſerted by the ball in its paſſage. It follows therefore, that a ball moving with this velocity is reſiſted by the whole preſſure of the atmoſphere, excluſive of the effects ariſing from the air diſplaced, and this additional reſiſtance muſt in moſt caſes be very great: for example, the diameter of an iron 24 pounder being 5.457 inches, we ſhall have the area of a great circle of the ball = 23.385 inches, and allowing 15 pounds avoirdupoiſe for the preſſure of the atmoſphere upon each ſquare inch, the reſiſtance to the ball's motion ariſing from the air's preſſure = 23.385 × 15 = 350.78, or a weight of 350.78 pounds: to this muſt be added the reſiſtance which is cauſed by the inertia of the air diſplaced: ſuppoſe the ball's velocity to be 1600 feet in a ſecond, and z = the ſpace through which a body muſt fall by gravity to acquire the velocity of 1600 feet in a ſecond, then will this ſpace $z = 39793$ feet; and becauſe the ball's diameter = 5.457 inches, the area of a great circle = .16240 parts of a foot, and the reſiſtance to the ball's motion being equal to the weight of a column of air,|| the altitude of which $= \frac{39793}{2}$ feet, and baſe = .16240 parts of a ſquare foot will be equal in weight to 3231 cubic feet of air, or 3755 ounces = 234.8 pounds: this being added to 350.7 before found, becomes 585.5 pounds for the whole reſiſtance ariſing from inertia

|| Sect. V. Prop. II. & Prop. IV.

inertia and preſſure. * Mr. Robins found by experiment, that the reſiſtance was about 540 pounds, which is leſs than that juſt determined by 44.5 pounds; this difference may be eaſily accounted for from the method adopted by Mr. Robins for determining the reſiſtance, which was by finding what reſiſtance was oppoſed to a ball of $\frac{3}{4}$ inch, moving with a given velocity, and inferring from thence what would be the reſiſtance to a ball of 5.457 inches diameter, if the velocity were nearly the ſame; by which it is manifeſt, that a ſmall error in the original experiment, for determining the leſſer reſiſtance, will cauſe a conſiderable variation in the greater reſiſtance, if it be inferred from the leſſer.

* Robins' Gunnery, Vol. I. p. 142.

There is another circumſtance alſo which tends ſtill to increaſe the reſiſtance to bodies moving with very great velocities, ſuch for inſtance as 1800 or 2000 feet in a ſecond. In this caſe the air being condenſed before the ball exerts a force of elaſticity againſt it in proportion to the compreſſion; and it is this repulſion which renders any augmentation of the velocities of military projectiles beyond 1200 or 1300 feet in a ſecond, of little uſe in increaſing either the horizontal range, or the effectual impetus of the bullet.

This elaſtic force which the air exerts againſt bodies of ſmall weight, but moving with conſiderable velocities, may become ſo great in proportion to the weight, as not only to deſtroy the motion communicated, but even to repel them; which is obſerved frequently to happen when very ſmall ſhot are diſcharged by a large quantity of powder; in which caſe the ſhot return back in a direction contrary to that in which they were projected: and it ſeems at firſt ſight ſingular, although ſtrictly coincident with the theory, that a ſmaller charge of powder by giving the ſhot leſs initial velocity will cauſe it to fly further than a greater charge, if the quantity of the charge ſhould exceed a certain limit.

VII.

Every thing remaining as in the laſt propoſition, it is required to determine the

Fig. XXVIII.

the time in which a body deſcribes any given ſpace co.

The velocity* at the point O $= \sqrt{4lb} \times e^{-\frac{3x}{8nd}}$. If therefore the † time required be T, we ſhall have $\dot{T} = \frac{\dot{x}\, e^{\frac{3x}{8nd}}}{\sqrt{4lb}}$, and the time required = fluent of $\frac{\dot{x}\, e^{\frac{3x}{8nd}}}{\sqrt{4lb}}$: to obtain the fluent let $e^{\frac{3x}{8nd}} = v$, wherefore $\frac{3x}{8nd} = \log. v$, and $\dot{x} = \frac{8nd}{3} \times \frac{\dot{v}}{v}$, wherefore $\dot{x}\, e^{\frac{3x}{8nd}} = \frac{8nd}{3} \times \dot{v}$, the fluent of which $= \frac{8nd}{3} \times v = \frac{8nd}{3} \times e^{\frac{3x}{8nd}}$, and the time required $= \frac{8nd}{3 \times \sqrt{4lb}} \times e^{\frac{3x}{8nd}}$; but this quantity ſhould vaniſh with x the ſpace deſcribed, but when $x = 0$, $\frac{8nd}{3 \times \sqrt{4lb}} \times e^{\frac{3x}{8nd}} = \frac{8nd}{3 \times \sqrt{4lb}}$: if therefore $\frac{8nd}{3 \times \sqrt{4lb}}$ be ſubtracted from the fluent juſt found, we ſhall have the time $T = \frac{8nd}{3\sqrt{4lb}} \times \overline{e^{\frac{3x}{8nd}} - 1}$. In theſe caſes the body is ſuppoſed to be acted on by no other force except thoſe of projection and reſiſtance, and conſequently moves with a velocity continually retarded; but if a body be impelled through a fluid by the force of gravity, this force compounded with that of reſiſtance may generate either a retarded or accelerated, or in ſome extreme caſes, as it will hereafter appear, an uniform velocity of motion: and if the body be acted on by no force out of the vertical direction in which gravity acts, that is, if it be either gradually accelerated from quieſcence, or be projected in a vertical line, the motion will be rectilinear.

* Sect. VI. Prop. VI.

† Sect. III. Prop. III.

VIII.

Fig:XXIX.

Let a ſpherical body deſcend in a fluid from reſt; having given the diameter of the ſphere and its ſpecific gravity, relatively to that of the fluid, it is required to aſſign the velocity of the ſphere at any given point o of the ſpace deſcribed A o.

Let the ‡ ſphere's diameter $= d$, the ſpecific gravity of the ſphere $= n$, that of the fluid $= 1$; alſo let the ſpace AO which the ſphere has deſcribed from and $= x$: ſince the ſphere's weight $= \frac{d^3 p.n}{b}$ and the weight of an equal bulk of the fluid $= \frac{d^3 p}{6}$, n being greater than 1, the abſolute or moving force whereby the ſphere endeavours to deſcend will be $\frac{d^3 p n}{6} - \frac{d^3 p}{6}$, which will be always the ſame, whatever be the velocity of the ſphere's motion (it being ſuppoſed always that the fluid is infinitely compreſſed): but the reſiſtance oppoſed on account of the ſphere's motion will depend on the velocity. Let z be the ſpace through which a body muſt fall by its gravity, ſo that it may acquire the velocity of the ſphere in the point O: then we ſhall have the reſiſtance oppoſed to the † ſphere equal to the weight $\frac{p z d^2}{8}$; the abſolute force therefore, whereby the ſphere endeavours to deſcend will be upon the whole $= \frac{d^3 p n}{6} - \frac{d^3 p}{6} - \frac{p z d^2}{8}$, and this weight being ‖ divided by the maſs moved will be the force, which accelerates the ſphere while it is deſcribing Oo; and ſince the weight of the ſphere is $= \frac{p.d^3 n}{6}$, we ſhall have the accelerating force $= 1 - \frac{1}{n} - \frac{3z}{4nd}$: and from the principles of * acceleration

‡ Euler. Comment. on Robins' Gunnery, Prop. II.

† Sect. V. Prop. IV. Cor.

‖ Sect. I. Prop. IX.

* Sect. III. Prop. V. and p. 56.

$\dot{z} = \overline{1 - \frac{1}{n} - \frac{3z}{4nd}} \times \dot{x}$, and by reduction $\frac{\dot{x}}{4nd} = \frac{\dot{z}}{4nd - 4d - 3z} = \frac{\dot{z}}{4d \times \overline{n-1} - 3z}$: to obtain the fluent, let $4d \times \overline{n-1} - 3z = v$: taking the fluxions, we have $3\dot{z} = -\dot{v}$, and $\frac{\dot{z}}{4d \times \overline{n-1} - 3z} = -\frac{\dot{v}}{3v}$, the fluent of which $= -\frac{1}{3} \times \text{log.}\ v = -\frac{1}{3} \times \text{log.}\ 4d \times \overline{n-1} - 3z$, wherefore $\frac{x}{4nd} = -\frac{1}{3} \times \text{log.}\ \overline{4d \times \overline{n-1} - 3z}$, which quantities ſhould vaniſh together, becauſe by the prob. when $z = 0$, $x = 0$: but when $z = 0$, $-\frac{1}{3} \times \text{log.}\ \overline{4d \times \overline{n-1} - 3z}$ becomes $-\frac{1}{3}\ \text{log.}\ 4d \times \overline{n-1}$; the entire fluent therefore will be $\frac{x}{4nd} = -\frac{1}{3}\ \text{log.}\ \frac{4d \times \overline{n-1} - 3z}{4d \times \overline{n-1}}$. Let $e = 2.71828$, &c. being the number the hyperbolic log. of which $= 1$, and we ſhall have $\frac{4d \times \overline{n-1} - 3z}{4d \times \overline{n-1}} = e^{-\frac{3x}{4nd}} = 1 - \frac{3z}{4d \times \overline{n-1}}$, and $\frac{3z}{4d \times \overline{n-1}} = 1 - e^{-\frac{3x}{4nd}}$, wherefore $z = \frac{4d}{3} \times \overline{n-1} \times \overline{1 - e^{-\frac{3x}{4nd}}}$: and if the velocity at O, be referred to the velocity which the earth's gravity generates in bodies each ſecond of the time wherein they fall, l being $= 16\frac{1}{12}$ feet or 193 inches, we ſhall have the velocity at $O =$ [*] $\sqrt{\frac{16ld \times \overline{n-1}}{3}} \times \sqrt{1 - e^{-\frac{3x}{4nd}}}$.

Cor. 1. If the ſpecific gravity of the globe be infinite, when compared with that of the fluid wherein it deſcends, that is, if n be increaſed ſine limite, the quantity $-\frac{3x}{4nd}$ will be evaneſcent, and $e^{-\frac{3x}{4nd}}$ will become =

1

[*] Sect. III. p. 28.

$1 - \frac{3x}{4nd}$, and $1 - e^{-\frac{3x}{4nd}} = \frac{3x}{4nd}$: wherefore the velocity will $= \sqrt{\frac{16ldn}{3} \times \frac{3x}{4nd}} = \sqrt{4lx}$, which is the velocity * acquired by a body deſcending from reſt in vacuo by the acceleration of gravity through the ſpace x.

* Sect. III. Prop. V.

Cor. 2. If the denſity of the globe is not increaſed ſine limite, but bears a conſiderable proportion to that of the fluid, the velocity acquired by the globe deſcending in the fluid through the ſpace x will approximate to the quantity $\sqrt{4lx} \times \overline{1 - \frac{3x}{16nd}}$, becauſe $e^{-\frac{3x}{4nd}} = 1 - \frac{3x}{4nd} + \frac{9x^2}{32n^2d^2}$ nearly, from which the approximation is eaſily deduced. For example, if a ſpherical body of the denſity of water deſcends in air from reſt through any ſpace x, let $V = \sqrt{4lx}$, be the ‡ velocity which the body would acquire by deſcending through the ſame ſpace in vacuo, then will the air's reſiſtance deſtroy that part of the whole velocity, ~~and~~ which is expreſſed by the fraction $\frac{3x}{16nd}$, that is, ſince in this caſe $n =$ about 860, the velocity loſt will be $= V \times \frac{3x}{13780\,d}$, which will be very near the truth when the quantity x, or the ſpace fallen through, exceeds not the diameter of the ball in a proportion of above 100 to 1.

‡ Sect. III. Prop. V. Cor. 3.

Cor. 3. Since $V = \sqrt{\frac{16dl \times \overline{n-1}}{3} \times \overline{1 - e^{-\frac{3x}{4nd}}}}$, if x be increaſed ſine limite, the quantity $e^{-\frac{3x}{4nd}}$ will approximate to and will ultimately $= 0$, wherefore the velocity acquired by the ſphere will continually approximate to $\sqrt{\frac{16ld \times \overline{n-1}}{3}}$, as a limit which it can never exceed, or even attain to in any finite time.

Cor. 4. Although a ſpherical body deſcending from reſt in a fluid, cannot attain to its mathematical limit, or maximum of velocity in any finite time, yet it will, in deſcending through a few diameters, approach ſo nearly to it, that the velocity will afterwards become to all obſervation uniform.

Thus a ſphere, the denſity of which is twice as great as that of the fluid wherein it deſcends, will in deſcribing 16 diameters only, arrive at its maximum of velocity within $\frac{1}{800}$ part of the whole; for if we put $x = 16d$, $n = 2$; then $\frac{3x}{4nd} = 6$, and $e^{-6} = \frac{1}{403}$; and $1 - e^{-6} = \frac{402}{403}$, and $\sqrt{1 - e^{-6}} = \frac{805}{806}$, and the velocity after the ſphere has deſcribed 16 diameters $= \sqrt{\frac{16ld \times \overline{n-1}}{3}} \times \frac{805}{806}$ being equal to the limit or greateſt velocity $\sqrt{\frac{16ld \times \overline{n-1}}{3}}$ within leſs than $\frac{1}{800}$ part of the whole, which difference, is far leſs than can be obſervable by the ſenſes; the ſphere therefore, after it has deſcribed the ſpace juſt mentioned, may be conſidered as moving uniformly.

IX.

Newt. Princip. Vol. II. Prop. XXXVIII. Cor. 2.

The greateſt velocity which can be acquired by a ſpherical body deſcending in a fluid is equal to that which would be acquired by it in deſcending from reſt in vacuo by the conſtant force of its comparative gravity, through a ſpace which is to $\frac{4}{3}$ of the diameter, as the denſity of the ſphere to the denſity of the fluid.

For when the ſphere by falling from reſt in a fluid medium has acquired its greateſt velocity, the increment of velocity is nothing, that is, the force of reſiſtance is equal to the body's weight, and conſequently the force of acceleration is nothing. Let therefore the denſity of the globe be to that of the fluid as $n : 1$, the globe's diameter $= d$, z the ſpace through which a body muſt fall from reſt by gravity to acquire the greateſt velocity above deſcribed, then we ſhall have the † accelerating force $= 1 - \frac{1}{n} - \frac{3z}{4nd}$, which by the problem muſt $= 0$; wherefore $\frac{3z}{4nd} + \frac{1}{n} = 1$, and $3z = 4d \times \overline{n - 1}$, and $z = \frac{4d \times \overline{n-1}}{3}$, equal to the ſpace due to the greateſt velocity acquirable by the ſphere; and if $l = 193$ inches, we have the greateſt *velocity $= \sqrt{\frac{16ld \times \overline{n-1}}{3}}$: but a ſpace which is to $\frac{4d}{3}$ as $n : 1$ becomes $\frac{4dn}{3}$, and if the weight of the ſphere in vacuo be Q, its comparative gravity whereby it endeavours to deſcend in the fluid $= Q - \frac{Q}{n}$, and the ‡ force whereby it is accelerated in its deſcent $= \frac{n-1}{n}$: and the velocity generated in the ſphere deſcending from reſt in vacuo through a ſpace $\frac{4nd}{3}$ by the ‖ force $\frac{n-1}{n} = \sqrt{4l \times \frac{4dn}{3} \times \frac{n-1}{n}}$ $= \sqrt{\frac{16ld\,\overline{n-1}}{3}}$; but this is the greateſt velocity which the ſphere can acquire in its deſcent by the preceding part of the ſolution.

† Sect. V. Prop. VIII.

* Sect. III. Prop. V. Cor. 3.

‡ Sect. I. Prop. IX.

‖ Sect. III. Prop. V. Cor. 3.

Or thus: Suppoſe it were required to aſſign how far the ſphere muſt fall from reſt in vacuo by the action of a conſtant

ſtant force, equal to that by which it endeavours to deſcend in the fluid, ſo that it may acquire the limit or greateſt velocity above deſcribed. The notation remaining as before, we have the force which accelerates the ſphere's deſcent $= \frac{n-1}{n}$, and V the greateſt velocity $= \sqrt{\frac{16ld \times \overline{n-1}}{3}}$: then ſince if S is the ſpace which a body deſcribes from reſt by the acceleration of any conſtant force F, while the velocity V is generated,* we have always $S = \frac{V^2}{4lF}$; applying this to the preſent caſe, $V =$ the greateſt velocity acquirable by the ſphere $=$ ‡$\sqrt{\frac{16ld \times \overline{n-1}}{3}}$, $F = \frac{n-1}{n}$, and $S = \frac{V^2}{4lF} = \frac{16ld \times \overline{n-1}}{3} \times \frac{n}{4 \times \overline{n-1} \times l} = \frac{4nd}{3}$, the ſpace required, which is to $\frac{4}{3}$ parts of the diameter as $n : 1$, or as the denſity of the ſphere to the denſity of the fluid.

|| Sect. III. Prop. V. Cor. 3.

‡ Sect. V. Prop. VIII. Cor. 3.

X.

Spherical bodies of very ſmall diameters deſcend in fluids ſpecifically lighter than themſelves, with nearly uniform velocities, which are in a direct ſubduplicate ratio of their diameters, the fluid wherein they deſcend being the ſame.

For the velocity* of any ſphere which deſcends in a fluid of half its ſpecific gravity, through a ſpace equal to 16 of its diameters, will approach to its limit or the greateſt acquirable velocity within $\frac{1}{800}$ part of the whole; wherefore, if the deſcent were continued for an unlimited ſpace, the velocity generated during this deſcent would not be more than

* Sect. V. Prop. VIII. Cor. 4.

than $\frac{1}{800}$ part of that which the body poſſeſſed after having deſcribed 16 diameters; wherefore the velocity after that time may be regarded, in a phyſical ſenſe, as abſolutely uniform: but while the diameter of a ſphere is diminiſhed, the ſpace of 16 diameters is diminiſhed in the ſame proportion; and when the diameter becomes evaneſcent, the ſphere will have deſcribed 16 diameters in the leaſt ſenſible portion of its deſcent: the velocity therefore of the deſcent will be as to obſervation entirely uniform. The ſame reaſoning may be extended to the deſcent of ſpherical bodies of any other ſpecific gravity, their diameters being diminiſhed ſine limite. Let $d =$ the diameter be expreſſed in inches, $n =$ the ſpecific gravity of the deſcending ſphere, that of the fluid being 1; moreover, let $l = 193$ inches, then we have the limit or greateſt velocity which the ſphere can acquire by deſcending in the † fluid = † Sect. V. Prop. IX. $\sqrt{\frac{16\,ld \times \overline{n-1}}{3}}$ inches in a ſecond; and by what has preceded,* this will be the uniform motion with which * Supra, ſpheres of evaneſcent diameters deſcend, and in the ſame fluid $\sqrt{\frac{16\,l\,\overline{n-1}}{3}}$ being conſtant, the velocities will be in a ſubduplicate ratio of the diameters.

Although it would be extremely difficult to reduce the propoſitions, relating to the velocities acquired by ſpherical bodies which deſcend in fluids to experimental trials, when the velocities are conſiderably accelerated; yet by obſerving the deſcent of ſmall ſpheres, the velocities of which are uniform, the theory admits of eaſy illuſtrations from matter of fact. The veſſel uſed for the purpoſe, was a cylinder of 6 inches in diameter and about 5 feet in height. In the firſt place, ſuch a veſſel being filled with water, if a number of ſmall ſpherical bodies of different diameters, but of the ſame ſpecific gravity, ſuch as very ſmall leaden ſhot, globules of mercury, &c. be at once ſuffered to deſcend from reſt through the water, the larger bodies will be obſerved to precede the other in their deſcent, and that in exact order, no ſmall body being ever ſeen deſcending ſo faſt as the larger ones, the velocity of deſcent alſo will be to all appearance uniform: theſe experiments ſo far correſpond with the theory, from which it has been deduced, that the uniform velocities with which ſmall ſpheres deſcend in fluids, are in a ſubduplicate ratio of their diameters. This is an illuſtration of a general kind, that which follows is more particular.

But it may be firſt noted that the diameter of a ſphere is obtained far more exactly from having given its ſpecific gravity and weight, than from direct menſuration. Let $d =$ the diameter of a ſphere in inches, alſo let n be its ſpecific gravity, that of water being $= 1$, and w the weight in grains determined by obſervation, then we ſhall always have $d = \left.\overline{\frac{w}{n}}\right|^{\frac{1}{3}} \times .19612$.

To apply this, the ſpecific gravity of a number of ſmall leaden ſhot was determined by weighing them in air and water, and found to be 11.1, and the weight of a ſingle ſhot ſelected from the reſt weighed 1.05 grains; wherefore to find the diameter, we have $n = 11.1$, $w = 1.05$ parts of a grain, and the diameter $= \left.\overline{\frac{1.05}{11.1}}\right|^{\frac{1}{3}} \times .19612$ $= .089$ parts of an inch: this ſphere was obſerved to deſcend in the veſſel of water 60 inches from reſt in ſomething more than two ſeconds, as nearly as could be determined. According to the approximated theory it ſhould have deſcribed almoſt 61 inches in the ſame time,

† Sect. V. Prop. IX. for according to this rule, the † velocity of deſcent $= \sqrt{\frac{16\,ld \times \overline{n-1}}{3}} = \sqrt{\frac{16 \times 193 \times .089 \times 10.1}{3}}$ $= 30.48$ inches in a ſecond. It will appear from an enſuing propoſition, in which the real time is inveſtigated wherein a ſphere deſcribes from reſt in a fluid any given ſpace by conſtant acceleration, how much the experiment, and this approximate value for the uniform velocity deviate from the truth.

In the mean time it may be remarked, that the experiment is ſufficient to ſhew the coincidence of the approximate theory with matter of fact, ſo as to juſtify the concluſions which will be deduced from this propoſition.

It appears therefore, that the velocities acquired by ſmall ſpherical bodies which deſcend in fluids are ſufficiently conſiſtent with the theory, when examined by ſuch trials as can be made on ſpheres, the magnitudes of which however ſmall, are yet ſubject to accurate and deciſive menſuration; we may therefore, without danger of error, extend this reaſoning to bodies whoſe minuteneſs renders ſuch real menſuration impoſſible. In all caſes it follows from the theory, that homogeneal ſpheres of the leaſt ſenſible magnitude will ‡ deſcend in a given fluid with velocities to all obſervation uniform; which velocities are in a ſubduplicate ratio of their diameters, and the ſmaller theſe diameters are, the more near will the approximation be to the theory. The ſame laws of deſcent will alſo obtain in evaneſcent particles, the figures of which are nearly ſimilar, although not ſpherical, but in various degrees; this too is confirmed by experiment, although in a leſs preciſe way than the preceding truths. For if a quantity of ſteel or braſs filings, or other minute homogeneal bodies of irregular ſhapes, be cauſed to deſcend from reſt in a fluid, you will always obſerve the largeſt bodies to precede the others in their deſcent: the velocities alſo with which theſe bodies deſcend, as in the former caſe, being uniform.

To apply theſe principles: It was obſerved that a ſphere of lead, the diameter of which = .089 parts of an inch, deſcended from reſt in water with a velocity of about 30 inches in a ſecond. If therefore the diameter of this leaden ſphere be diminiſhed in the proportion of $n : 1$, the velocity of deſcent * will be only the $\frac{1}{\sqrt{n}}$ part of the former velocity: thus if the diameter be diminiſhed in the ratio of a million to one, that is, if the diameter = .000000089 parts of an inch, the velocity of deſcent will be = .03 parts of an inch in a ſecond; and if the diameter be again diminiſhed in the ſame proportion, that is, of 1000000 : 1, the velocity of the particles' deſcent will be only .00003 in a ſecond, or about $\frac{1}{10}$ of an inch in an hour, allowing no additional retardation for the effects of tenacity or coheſion. To apply this reaſoning, we obſerve that if the parts of ſome ſubſtances, having been ſeparated by diſſolving menſtrua, are ſhaken together, ſo as to be diffuſed uniformly over the whole, they will continue for ſome time apparently quieſcent, and at length will be obſerved to deſcend equably; and from what has preceded, with a greater or leſs velocity, according to their magnitude and denſity, in reference to that of medium. Thus ſhould the diameters of the particles be equal to .0890 parts of an inch diminiſhed in a duplicate ratio of 1000000 to 1, and their ſpecific gravity 11.1 times greater than that of the fluid, the velocity of

‡ Prop. X. hujus.

* Prop. X. hujus.

their uniform deſcent will be that of about $\frac{1}{10}$ of an inch in an hour, provided the particles be ſpherical, or not greatly deviating from that figure. But the diminution of the parts of ſolid bodies diſſolved in fluids is not limited to that juſt deſcribed, either by obſervation or any method of argument hitherto made uſe of. On the contrary, arguments, ſuch as the nature of the ſubject admits, will induce us to conclude, that ſince matter conſidered as occupying ſpace in a metaphyſical ſenſe, is diviſible ſine limite, ſo the actual diviſion of material ſubſtance which is effected by natural cauſes is unlimited, as to our power of eſtimating it by obſervation; there is certainly no ſubdiviſion which can be expreſſed by numbers, but what we may reaſonably ſuppoſe to be exceeded in the operations of nature. We may therefore juſtly imagine the magnitude of the particles of ſome ſubſtances exiſting in a ſtate of ſolution, to be far leſs than that of the particles which deſcend in water with a velocity of .108 parts of an inch in an hour, and that in any aſſignable ratio: if the diameters are leſs than that above mentioned in the ratio of 1000000 : 1, the velocity of this deſcent will be diminiſhed in the ratio of 1000 : 1, and will be therefore equal to that of $\frac{1}{10}$ part of an inch in 1000 hours, being diminiſhed in a ſubduplicate ratio of the particles' diameters; wherefore the particles would be above a year in deſcending through one inch in the fluid; and it is obvious, that the magnitude of the particles may be ſtill diminiſhed, ſo that the deſcent ſhall not be ſenſible in any aſſignable time. This concluſion will be true, without making any allowance for the effects of friction, coheſion, and tenacity, which exiſt in all fluids in a greater or leſs degree, although in water and mercury, termed perfect fluids, they are ſcarcely if at all ſenſible in experiment: but in other fluids, a reſiſtance to the deſcent of bodies will be oppoſed by the tenacity, as well as the inertia of their parts.

It will follow upon the whole, that if the diameter of diſſolved particles deſcending with a ſlow and uniform motion in fluids void of tenacity and friction be known, and their denſity with that of the fluid be given, the velocity of their deſcent may be aſcertained a priori, and converſely, if their velocity of deſcent be obſerved, and their denſity with that of the fluid be known, the diameter of the particles may be deduced, provided they are ſpherical, or of other given figure, not deviating greatly from that of a ſphere. Let v be the parts of an inch which theſe particles deſcribe in their deſcent in one ſecond, alſo let the denſity of the particles be to that of the fluid as $n : 1$, then will the diameter of the ſpheres $= \frac{3v^2}{16l \times \overline{n-1}}$ parts of an inch, l being $= 193$ inches.

Thus, ſuppoſe particles of copper diſſolved in ſpirit of nitre, were obſerved to ſubſide one inch in .48 parts of an hour, that is, $\frac{1}{1728}$ part of an inch in a ſecond: and ſuppoſing the particles to be ſpherical, let it be required to determine their diameter: here referring to the laſt article, we have $v = \frac{1}{1728}$, $l = 193$, and if the ſpecific gravity of copper be to that of aqua fortis, as $9 : 1.3$, we ſhall have $n = \frac{9}{1.3} = 6.9231$, and the diameter of the particles $= \frac{3}{1728^2 \times 16 \times 193 \times 5.9231} =$ leſs than $\frac{1}{18205000000}$ part of an inch.

XI.

Let a ſperical body deſcend in a fluid from reſt at A, having given the diameter of the ſphere and its ſpecific gravity relatively to that of the fluid; it is required to aſſign the time in which the ſphere deſcribes any ſpace A O. Fig. XXIX.

The notation of the eighth propoſition remaining, ſince the velocity † at any point O $= \sqrt{\frac{16ld \times \overline{n-1}}{3}}$ † VIII. huj.

×

$\times \sqrt{1 - e^{-\frac{3x}{4nd}}}$, if T be the time of motion,* we ſhall have $\dot{T} = \sqrt{\frac{3}{16ld \times \overline{n-1}}} \times \frac{\dot{x}}{\sqrt{1 - e^{-\frac{3x}{4nd}}}}$, and the time of deſcribing $AO = \sqrt{\frac{3}{16 \times ld \times \overline{n-1}}} \times$ fluent of $\frac{\dot{x}}{\sqrt{1 - e^{-\frac{3x}{4nd}}}}$: to obtain the fluent, let $\sqrt{1 - e^{-\frac{3x}{4nd}}} = v$, wherefore $1 - e^{-\frac{3x}{4nd}} = v^2$, and taking the fluxions $\frac{3\dot{x}}{4nd} \times e^{-\frac{3x}{4nd}} = 2v\dot{v}$, and ſince $1 - v^2 = e^{-\frac{3x}{4nd}}$, multiplying one ſide by $\frac{4nd}{3e^{-\frac{3x}{4nd}}}$, and the other by its equal $\frac{4nd}{3 \times \overline{1 - v^2}}$, we have $\dot{x} = \frac{8v\dot{v}nd}{3 \times \overline{1 - v^2}}$, and $\frac{\dot{x}}{\sqrt{1 - e^{-\frac{3x}{4nd}}}} = \frac{8nd}{3} \times \frac{\dot{v}}{1 - v^2}$: the fluent of $\frac{\dot{v}}{1 - v^2}$ is $= \frac{1}{2} \times$ hyperbolic logarithm of $\frac{1 + v}{1 - v}$: the fluent therefore of $\frac{\dot{x}}{1 - e^{-\frac{3x}{4nd}}}$ $=$ the fluent of $\frac{8nd}{3} \times \frac{\dot{v}}{1 - v^2} = \frac{8nd}{3 \times 2} \times \log. \frac{1 + v}{1 - v}$ $= \frac{4nd}{3} \times \log. \frac{1 + \sqrt{1 - e^{-\frac{3x}{4nd}}}}{1 - \sqrt{1 - e^{-\frac{3x}{4nd}}}}$; and the time required $= \sqrt{\frac{3}{16ld \times \overline{n-1}}} \times$ fluent of $\frac{\dot{x}}{\sqrt{1 - e^{-\frac{3x}{4nd}}}}$ $= \sqrt{\frac{3}{16ld \times \overline{n-1}}} \times \frac{4nd}{3} \times \log. \frac{1 + \sqrt{1 - e^{-\frac{3x}{4nd}}}}{1 - \sqrt{1 - e^{-\frac{3x}{4nd}}}}$ $=$

* Sect. III. Prop. III.

$$=\sqrt{\frac{n^2 d}{3l \times \overline{n-1}}} \times \log.\ \frac{1+\sqrt{1-e^{-\frac{3x}{4nd}}}}{1-\sqrt{1-e^{-\frac{3x}{4nd}}}}\ \text{ſeconds.}$$

Cor. 1. When n is increaſed ſine limite the ſphere's motion will not be retarded by the medium through which it deſcends, in this caſe $e^{-\frac{3x}{4nd}}$ approximates to $1-\frac{3x}{4nd}$,

and log. $\frac{1+\sqrt{1-e^{-\frac{3x}{4nd}}}}{1-\sqrt{1-e^{-\frac{3x}{4nd}}}} = \log.\ \frac{1+\sqrt{\frac{3x}{4nd}}}{1-\sqrt{\frac{3x}{4nd}}} =$

$2 \times \sqrt{\frac{3x}{4nd}}$: the time of deſcribing any ſpace x therefore will be $\sqrt{\frac{n^2 d}{3l \times \overline{n-1}}} \times \sqrt{\frac{3x}{nd}} = \sqrt{\frac{x}{l}}$ ſeconds, which appears alſo from Sect. III. Prop. IV.

The times wherein ſpherical bodies deſcend from reſt in fluids, determined by theory, admit of very exact experimental proof. In order to render trials of this ſort capable of variety, a hollow braſs ſphere was conſtructed, which by means of an aperture could be ſo loaded with leaden ſhot, ſand, &c. that its ſpecific gravity might be altered in any aſſigned ratio; the aperture being cloſed, the whole figure was perfectly ſpherical: that the diameter might be aſcertained, and the ſpecific gravity be afterwards fixed for the experiment with greater certainty, the ſphere was ſo adjuſted by incloſed weights, that it would reſt in water perfectly quieſcent wherever it was placed: after this adjuſtment, the weight was found to be = 1093 grains, and ſince the ſpecific gravity was the ſame as that of the water wherein it was immerſed, referring to the preceding rule,† we ſhall have the ſphere's diameter $= \overline{1093}^{\frac{1}{3}} \times .19612 = 2.0202$ inches: a weight of 273 grains being incloſed in the ſphere when adjuſted in the manner above deſcribed, the whole weight muſt now be 1366, and its ſpecific gravity to that of water, as 1366 to 1093, that is, as 1.25 : 1; upon letting this ſphere deſcend in water from reſt, it was obſerved that the time of deſcribing 60 inches was about 3 ſeconds.

To compare this experiment with the theory, we have the diameter of the ſphere $d = 2.0202$ inches,

† Page 146.

$x = 60$, $n = 1.25$, and $\frac{3x}{4nd} = \frac{180}{10.101} = 17.82$, and

$$\frac{1 + \sqrt{1 - e^{-\frac{3x}{4nd}}}}{1 - \sqrt{1 - e^{-\frac{3x}{4nd}}}} = 219360000,$$ the logarithm of which from the tables $= 8.3411574$, which being multiplied by 2.3015850, will become 19.206, the hyperbolic logarithm of $\frac{1 + \sqrt{1 - e^{-\frac{3x}{4nd}}}}{1 - \sqrt{1 - e^{-\frac{3x}{4nd}}}}$: we have alſo

$$\sqrt{\frac{n^2 d}{3l \times n - 1}} = \sqrt{\frac{1.25^2 \times 2.0202}{3 \times 193 \times .25}} = .14767,$$ and the time of deſcent by theory $= \sqrt{\frac{n^2 d}{3l \times n - 1}} \times \log.$

$$\frac{1 + \sqrt{1 - e^{-\frac{3x}{4nd}}}}{1 - \sqrt{1 - e^{-\frac{3x}{4nd}}}} = 19.206 \times .14767 = 2.83$$ ſeconds. The ſphere deſcended upon trial in 3 ſeconds, which deviates from the theory $\frac{17}{100}$ parts of a ſecond.

A few particulars relative to the method of making theſe experiments accurately may be inſerted. The ſphere ſhould be very ſmooth, in order that no diminution of motion may ariſe from friction.

2. That the ſphere may begin its deſcent at a definite inſtant of time, it ſhould be ſuſpended by a very fine ſilk line, ſo as to be juſt under the water's ſurface: if the line be divided with ſciſſars at the beat of a pendulum clock, the beginning of the deſcent will be eſtimated without error.

3. The weight of the ſphere ſhould be ſo diſpoſed, that the centre of gravity may be in the ſame vertical line with the centre of the ſphere and the point from which the ſphere is ſuſpended, and ſhould be ſomewhere near the loweſt ſurface of the ſphere when ſuſpended.

The want of due attention to theſe particulars has occaſioned the reſults of ſome experiments, which have been made on the deſcent of bodies in fluids, to deviate conſiderably from the theory, whereas the proper precaution in conſtructing the experiment being obſerved, the theory will be found to agree with matter of fact to a very ſatisfactory degree of preciſion.

By altering the quantity of weight incloſed within the ſphere, the times of deſcent may be varied in any ratio. The enſuing table will ſhew the reſults of ſeveral experiments made by incloſing different weights within the ſphere, after it had been adjuſted, ſo as to remain in water quieſcent wherever it was placed, in which caſe it has already been ſhewn, that the ſphere's † weight = 1093 grains, and the diameter = 2.0202 inches.

† Page 151.

No. I.

The weight incloſed was 11 grains, and conſequently n or the ſphere's ſpecific gravity was $\frac{1104}{1093}$, that of water being = 1.

Experiment	Time of deſcribing from reſt 60 inches in water by experiment.	Mean time of deſcent.	Time by theory.	Error of experim.
1	$14\frac{3}{4}$			
2	$14\frac{1}{2}$	14.6	13.94	+ .66
3	$14\frac{3}{4}$			

No. II.

The weight incloſed within the ſphere was 3 grains, and conſequently n, or the ſphere's ſpecific gravity in reference to water = $\frac{1096}{1093}$, that of water being = 1.

Experiment	Time of deſcribing from reſt by experiment 60 inches.	Mean of theſe obſervations.	Time by theory.	Error of experim.
1	$27.\frac{1}{2}$			
2	27	27.3	26.66	+ .67
3	$27\frac{1}{2}$			

No. III.

The weight incloſed in the ſphere was 1 grain, and conſequently the ſphere's ſpecific gravity $= \frac{1094}{1093}$, that of water being $= 1$.

Experiment.	Time of deſcribing 60 inches from reſt by experiment.	Mean of theſe obſervations.	Time by theory.	Error of experim.
1	45			
2	44 $\frac{1}{4}$	44·33	46.21	— 1.88
3	43· $\frac{3}{4}$			

Theſe experiments are coincident with the computed times to a ſufficient degree of exactneſs, conſidering the extreme nicety requiſite to conſtruct them ſo as to be conſiſtent with the conditions required by the theory: for it is almoſt impoſſible to prevent ſome ſmall degree of oſcillation in the ſphere while it deſcends; it is alſo equally difficult to make the figure ſo perfect that the deſcent may be preciſely in a vertical line. It is alſo to be noted, that when the incloſed weight, which may properly in this caſe be termed the moving force, is very ſmall, as in the laſt ſet of experiments, it is very difficult to ſay that it ſhall not vary $\frac{1}{40}$ or $\frac{1}{50}$ part of the whole from the truth: for in this caſe, the unavoidable errors (however ſmall) in the conſtruction of weights may conſpire with thoſe of balancing the ſphere in the water previous to the experiments. Theſe cauſes are certainly ſufficient to create ſome aberration of the time by experiment from that which is deduced from computation, no greater diſagreement however is obſerved between the theory and matter of fact, than what ſhould be rather attributed to thoſe cauſes than to any imperfection in the theory.

When a body is immerſed in a fluid, it will either * aſcend or deſcend according as its weight is leſs or greater than that of an equal bulk of the fluid: in either caſe the body may be conſidered as acted on by three forces, one

* Cotes's Hydroſtatics, Lect. 4.

whereof is its weight or natural gravity, which is diminiſhed by the weight of an equal bulk of the fluid; theſe two forces always act in a direction oppoſite to each other, and are invariable in quantity, whatever be the velocity with which the ſphere moves. To theſe muſt be added a third, i. e. that of reſiſtance, which will depend both in direction and quantity on the ſphere's motion; if the ſphere aſcends, the force of reſiſtance conſpires with that of gravity, but the ſum of both forces is leſs than the weight of a quantity of fluid equal in magnitude to the ſolid, for which reaſon, the ſphere will continue to aſcend in the direction of the ſtronger impulſe.

In the preceding propoſitions the ſpherical bodies have been ſuppoſed to deſcend in the fluid, and from the laws of this deſcent it has appeared, that bodies of the greateſt known ſpecific gravity,‡ if they be divided into ſufficiently minute parts by the menſtrua wherein they are diſſolved, may remain ſuſpended in them for any length of time. In this manner will aqua regia or even ether hold ſuſpended the parts of that moſt ponderous of all metals gold: and in the ſame manner air will retain water ſuſpended in it, when once ſubdivided into ſufficiently minute parts; and various ſimilar inſtances will naturally ſuggeſt themſelves. But another difficulty attends theſe phænomena which ſhould be here taken notice of.

‡ Page 148.

Although the laws of reſiſtance will ſufficiently explain the reaſon why bodies when diſſolved remain ſuſpended in fluids ſpecifically lighter than themſelves, yet it does not follow from any thing that has preceded, how a body divided into parts however minute, can poſſibly aſcend in a fluid ſpecifically lighter than itſelf; whereas it is well known that in ſome ſolutions, when the ſolid is to be diſſolved is placed at the bottom of a veſſel into which the diſſolving fluid is poured, the parts of the ſolid during the ſolution, without any motion whatever being communicated to the veſſel, will be diffuſed throughout the ſubſtance of the diſſolving fluid, appearing at firſt ſight to overcome the natural tendency of all bodies toward the earth's centre, and to have ſome new power of aſcent impreſſed upon it in conſequence of the diſunion of its parts. How far this is in reality the caſe may probably appear from the ſubſequent propoſitions and arguments deduced from them.

XII.

If a ſpherical body of a given diameter be immerſed in a fluid, and if the ſphere's weight bears no ſenſible proportion to the weight of an equal bulk of the fluid, the velocity of its aſcent will be uniform from the firſt inſtant of its motion, and will be the ſame as that which a heavy body acquires in falling from reſt by the acceleration of gravity through $\frac{4}{3}$ of its diameter.

Fig. XXX.

Let the aſcent of the ſphere begin from quieſcence at A, and let $AO = x$: moreover, let z be the ſpace through which a body muſt fall from reſt by the acceleration of gravity, ſo as to acquire the velocity in O. Then will the reſiſtance ‡ to the ſphere's motion at O = the weight $\frac{pd^2z}{8}$, d being the ſphere's diameter, and $p = 3.14159$, &c. the ſphere's weight will alſo reſiſt the aſcent, and if n be a very ſmall fraction repreſenting the ſpecific gravity of the ſphere, its weight will be expreſſed by $\frac{d^3pn}{6}$, wherefore $\frac{pd^3n}{6} + \frac{pd^2z}{8}$ will be the whole force of reſiſtance oppoſed to the ſphere's aſcent: but the force whereby the ſphere endeavours to aſcend is the weight of a quantity of the fluid equal in bulk to the ſphere, which weight $= \frac{pd^3}{6}$, becauſe the ſpecific gravity of the fluid is $= 1$; wherefore the force which upon the whole impels the ſphere in its aſcent $= \frac{pd^3}{6} - \frac{pd^3n}{6} - \frac{pd^2z}{8}$, and this being divided by the * ſphere's weight $\frac{pd^3n}{6}$, will give the force whereby

‡ Sect. V. Prop. IV.

* Sect. I. Prop. IX.

by the ſphere is accelerated at O, which will therefore be $\frac{1}{n} - 1 - \frac{3z}{4nd}$; or ſince 1 vaniſhes in reſpect of $\frac{1}{n} - \frac{3z}{4nd}$, the force of acceleration will be $\frac{1}{n} - \frac{3z}{4nd}$, and from the principles before ‖ demonſtrated, we have $\overline{\frac{1}{n} - \frac{3z}{4nd}} \times \dot{x} = \dot{z}$, or $\frac{\overline{4d - 3z}}{4nd} \times \dot{x} = \dot{z}$, and $\frac{\dot{x}}{4nd} = \frac{\dot{z}}{4d - 3z}$, wherefore $\frac{x}{4nd} = -\frac{1}{3} \log. \overline{4d - 3z}$ being the variable part of the fluent, to which if the conſtant part $+\frac{1}{3} \log. 4d$ be added, we ſhall have $\frac{x}{4nd} = -\frac{1}{3} \times \log. \frac{4d - 3z}{4d}$. Let $e = 2.71828$, being the number the hyperbolic logarithm of which is equal to 1; this will give $\frac{4d - 3z}{4d} = e^{-\frac{3x}{4nd}} = 0$, (becauſe $\frac{3x}{4na}$ is infinite,) and conſequently $4d - 3z = 0$, or $z = \frac{4d}{3}$, and if $l = 193$ inches, the * velocity at $O = \sqrt{\frac{16ld}{3}}$ inches in a ſecond, which is the ſame velocity which would be acquired by a heavy body which falls by the acceleration of gravity from reſt through a ſpace equal to $\frac{4}{3}$ of the ſphere's diameter. Since this reaſoning is applicable without the leaſt alteration to any other point Q, in which caſe the velocity at Q will come out the ſame as before $= \sqrt{\frac{16ld}{3}}$ inches in a ſecond, it follows, that the velocity of the ſphere's aſcent is entirely uniform.

‖ Sect. III. p. 28.

* Sect. III. Prop. V.

Cor. 1. A globe without weight aſcends in fluids of any different denſities with the ſame uniform velocity, provided the diameter of the globe be unaltered.

Cor. 2. Globes without weight of different diameters aſcend in fluids with uniform velocities, which are in a direct ſubduplicate ratio of their diameters.

Cor. 3. The concluſions deduced from this propoſition, follow alſo from Prop. VIII. where the velocity of the ſphere's deſcent is ſhewn to be $V = \sqrt{\frac{16\,l\,d \times \overline{n-1}}{3}} \times \sqrt{1 - e^{-\frac{3x}{4nd}}}$; n being aſſumed greater than 1. If n be aſſumed leſs than 1, ſo that the ſphere may aſcend in the fluid, the inveſtigation will be in every ſtep the ſame as before, except as far as regards the quantity $n - 1$, which will become $1 - n$; ſo that the velocity of the ſphere's aſcent will be defined by the equation $V = \sqrt{\frac{16\,l\,d \times \overline{1-n}}{3}} \times \sqrt{1 - e^{-\frac{3x}{4nd}}}$; and if $n = 0$, as in the propoſition juſt demonſtrated, $e^{-\frac{3x}{4nd}} = 0$, and $V = \sqrt{\frac{16\,l\,d}{3}}$ as before.

Cor. 4. If the ſphere ſhould not be abſolutely without weight when compared with the weight of an equal bulk of the fluid wherein it is immerſed, yet the preceding principles will be ſufficiently true for any practical application of them to matter of fact, provided the ſpecific gravity of the ſpheres be not in a greater proportion to that of the fluid, than of about 1 to 100: for in this caſe, the error of the rules for determining the velocity of aſcent will not exceed $\frac{1}{200}$ part of the true velocity, which would not be ſenſible in obſervation: and in general, if the ſphere's ſpecific gravity be a ſmall fraction n, that of the fluid wherein it is immerſed being $= 1$, the error of the preceding rule ariſing from the ſphere's weight, will be that part of the true velocity which is expreſſed by the fraction $\frac{n}{2}$, which appears from inſpecting the rule in Cor. 3.

Theſe concluſions ſeem to be related to ſeveral natural phenomena, the ſolution of which from mathematical principles requires ſome conſiderations to be premiſed.

Although it is certain that no material ſubſtance is deſtitute of gravity, yet it is equally true that there are fluids in nature of ſo rare and ſubtle a texture, that their weight is inſenſible in regard to any diſcoverable effects produced by it in ſome natural phænomena.

Among theſe phænomena muſt be reckoned the minute bubbles or ſpherical cavities, which aſcend in fluids during fermentations, ſolutions, &c. Theſe ſmall cavities are filled with airs or elaſtic vapours of different kinds, ſome of which are ſpecifically heavier than common air, and ſome far lighter than that fluid. Others again conſiſt of a ſpecies of vapour ſo rare that its weight is ſcarcely ſenſible: but whatever be the quality of theſe vapours, their weight affects not perceptibly the velocity of the aſcending ſpherical cavities containing it, which appears from Cor. 4. huj. according to which, if the ſpecific gravity of the included vapour were only 600 times leſs than that of the fluid wherein the ſpherules aſcend, the variation of the velocity occaſioned by the weight, will be no more than $\frac{1}{1200}$ part of the whole, which is intirely inſenſible in experiment.

That the contents of all theſe aerial ſpherules are elaſtic is manifeſt from their cavities admiting of dilatation and compreſſion. For it is evident to the ſenſes, that heat or a diminution of external preſſure, will cauſe them to expand into larger portions of ſpace, whereas the application of cold or an additional preſſure will create a diminution of their bulk: and if the temperature of the fluid wherein they aſcend and the external preſſure remain the ſame, their diameters will be unaltered.

The laws according to which a ſpherical body without weight aſcends in any fluid, demonſtrated in Prop. XII. huj. are applicable to the aſcent of the minute air bubbles above mentioned, although the figure of a ſpherical bubble will be changed during its aſcent in a fluid if the diameter be conſiderable; yet when the ſpheres containing theſe vapours are very ſmall, no ſuch alteration in their figure takes place, which may be known, not only from their apparent perfectly globular form, but from their equable and vertical aſcent: whereas when the diameter is ſo great that the figure ſuffers alteration, the ſphere's motion is neither equable nor in a vertical line, but in various irregular directions, tending however upon the whole conſtantly upward. It muſt alſo be added, that although theſe ſpherules being leſs compreſſed as they approach the ſurface of the fluid, the elaſticity of their contents will neceſſarily cauſe an increaſe of their diameters, yet this increaſe will be ſo ſmall in moſt caſes, that the velocities as determined by Prop. XII. will not be ſenſibly affected thereby.

To

Fig. XXX. To illuſtrate this, let *FCE* repreſent the ſurface of a fluid of the ſame denſity with water, *B* a ſmall aerial ſphere aſcending in the vertical line *BC*; produce *BC* to *A*, ſo that *CA* may be equal to 34 feet: now it is manifeſt, that the preſſure upon the ſphere at *B* will be as the ſpace *BA*, and conſequently if the depth *BC* ſhould be one foot, the different preſſures at *B* and *C* will be as *BA* : *CA*, or as 34 : 33; and the diameter of the ſphere at *B* will be to its
‡ Vid. infra p. 163. diameter at *C* in an ‡ inverſe ſubtriplicate ratio of 34 : 33, or as 101 : 102 nearly: which proportion of inequality is ſo ſmall, that the diameter of the ſphere may juſtly be regarded as conſtant during the whole of its paſſage from *B* to *C*. This being the caſe, the velocity of the ſphere's aſcent may be eſtimated from having given its diameter,
† Sect. V. Prop. XII. by referring to the preceding † Prop. in which it is ſhewn that the velocity of the aſcending ſpherule is intirely uniform, and equal to that which a heavy body acquires in deſcending from reſt by the acceleration of gravity through a ſpace equal to $\frac{4}{3}$ of the ſphere's diameter. So that if $d =$ the ſphere's diameter, v the velocity of its aſcent, $l = 193$ inches, it has been ſhewn, that $v = \sqrt{\frac{16dl}{3}}$, and conſequently $d = \frac{3v^2}{16l}$. To apply this, ſuppoſe it were obſerved that a ſmall air bubble aſcended in water at the rate of three inches in a ſecond, and let it be required to determine its diameter; here $v = 3$, and by the rule $d = \frac{3v^2}{16l} = \frac{3 \times 9}{16 \times 193} = \frac{1}{114}$ part of an inch.

This ſolution is ſufficiently exact for determining the aſcent of aerial ſpherules in water and in fluids of ſimilar denſity, (as long as the atmoſphere preſſes upon their ſurface,) through a ſmall ſpace. The proportion of *BA* to *CA* is increaſed either by leſſening or removing the preſſure of the atmoſphere, by increaſing the ſpace *BC* through which the ſphere aſcends, or by cauſing the ſphere to aſcend in a fluid of great denſity compared with that of water, ſuch as mercury: for in this latter caſe, the height *AC* = 34 feet muſt be diminiſhed in the proportion of 14 : 1,
Fig. XXX. ſo as to become 2.44 feet: let this = *CD*, if therefore the ſpherule aſcends through *BC* = 1 foot, the preſſures at *B* and *C* would be as 3.44 : 2.44, or as *BD* : *CD*, and the diameters at *B* and *C* in a ſubtriplicate ratio of 2.44 : 3.44;

which is too great a proportion of inequality to admit of the diameters being regarded as conſtant. In theſe caſes recourſe muſt be had to a ſubſequent propoſition in order to determine the velocities and times of aſcent.

It is next to be conſidered in what manner theſe aerial bubbles act during fermentations, ſolutions, &c.

XIII.

When a ſolid body is immerſed in a fluid which diſſolves it, the parts of the diſſolved ſubſtance although of greater ſpecific gravity than the menſtruum, are yet elevated ſo as to be diffuſed in ſome degree over the whole maſs, and this elevation of the particles is cauſed by the globules of vapour which eſcape from the ſolid during its ſolution.

The diſcoveries of Dr. Black, Sir J. Pringle, Dr. Macbride, &c. relative to certain airs or vapours exiſting within the ſubſtance of bodies in a fixed or latent ſtate, ſeem to make it evident that the power whereby the parts of bodies cohere, ariſes from the action of theſe vapours, or as they have been denominated, aerial fluids.

On this account fixed † air has been termed the cementing principle, and although other cauſes may poſſibly concur in the production of the ſolidity of bodies' textures, yet it follows indubitably from the admirable experiments of the gentlemen juſt mentioned, that when the fixed air which naturally exiſts within the pores of a ſubſtance is abſtracted from it, the coheſion of its parts is at the ſame time deſtroyed; and the principle is ſtill rendered more obvious from reſtoring the cementing power or fixed air, whereby in many caſes a new coheſion of parts takes place, cauſing the ſeparated particles of the body to reunite into a ſolid maſs. Thus, if animal fleſh be im-

† Macbride's Eſſays.

merſed under water, after ſome time ſmall bubbles of air will be obſerved to riſe in the water, a ſign that fermentation has begun to diſſolve the texture of the fleſh: which by this proceſs being deprived of its natural quantity of fixed air, loſes its coheſive firmneſs of texture and becomes putrid: but if a ſufficient quantity of fixed air be reſtored to it, the putrid fleſh will recover its ſolidity of texture, and become as ſweet to the ſmell as before. What more particularly belongs to the ſubject of the propoſition, is an appearance always obſerved during the fermentative proceſs juſt mentioned; as ſoon as the air bubbles begin to ariſe, ſmall particles of the fleſh are likewiſe detached from the putreſcent ſubſtance, and aſcend to the top of the water; and this aſcent is cauſed by the bubbles of fixed air which aſcend during the fermentation. Each of theſe bubbles would if unimpeded riſe with a velocity which a heavy body acquires in deſcending from reſt by the acceleration of gravity * through a ſpace equal to $\frac{4}{3}$ of the ſphere's diameter: but if any ſolid ſubſtance be attached to it, the ſphere will nevertheleſs aſcend, although more ſlowly, provided the weight of the attached particle exceeds not the weight of a quantity of the fluid equal in bulk to itſelf and the ſphere taken together; and this will be true whatever be the ſpecific gravity of the particle which is elevated by the aſcending bubble of air.

In like manner during ſolutions of ſaline, metalline, or other bodies, vapours of various kinds are obſerved to ariſe in ſmall bubbles, and to eſcape from the ſurface of the diſſolving fluid; and it is not improbable, that however theſe vapours may differ in poſſeſſing ſpecific properties peculiar to each, yet that of cauſing or contributing to the coherence of the parts of bodies may be a property common to them all: however this may be, it ſeems manifeſt, that the numerous aerial ſpherules which aſcend from any ſubſtance immerſed in a diſſolving fluid, carry up with them the diſunited parts of the ſolid, and in greater or leſs quantity, according to the magnitude of the parts and their denſity in reference to that of the diſſolving medium. An inſtance of this kind may be eaſily obſerved by immerſing a piece of ſugar in a glaſs veſſel of water, the air bubbles during their aſcent viſibly carrying up with them long ſtreams of the diſſolved ſugar.

In what manner the diſunited parts of bodies of the greateſt ſpecific gravity when once elevated in menſtru-

* Sect. V. Prop. XII.

ums, are notwithſtanding their denſity there held ſuſpended, has been already deſcribed. † † Page 148.

The diameters of the ſpherules of air which aſcend during fermentations, ſolutions, &c. have been hitherto conſidered as invariable, which will be nearly true, provided they continue to ſuffer invariable compreſſion, and the temperature of the fluid continues the ſame. Although it has been ſhewn, that during a ſphere's ‡ aſcent through ‡ Page 160. any ſmall ſpace, the compreſſion alters not very conſiderably, as long as the atmoſphere continues to preſs on the ſurface of the fluid, yet when that preſſure is either removed or much diminiſhed; or if the fluid ſhould be expoſed to various degrees of heat, the diameters of the aſcending bubbles will vary in ſo great a degree that the velocity of aſcent will no longer be deducible from the rules already demonſtrated. In order to inveſtigate the velocity with which a ſpherule of air or vapour aſcends, allowing for the variation of the diameter, the effects of heat are not conſidered, it being impoſſible to reduce them to geometrical eſtimation. The temperature of the fluid therefore in which the aerial ſpherules ariſe being the ſame, the only alteration in their magnitudes and diameters will be occaſioned by the variable preſſure acting upon them at different diſtances from the ſurface of the fluid: this alteration of the diameter will depend on the law, which the ſpaces occupied by the elaſtic vapour obſerve in reference to the reſpective preſſures: and it is * found by ſome experiments which have been made on the ſubject (although not yet ſufficiently extenſive to entirely eſtabliſh the propoſition, but which muſt at preſent be allowed as true in order to proceed with the argument) that the elaſtic forces exerted by a given quantity of this vapour, will be greater in the ſame proportion as the ſpaces occupied by it are leſs: and it is evident that the elaſtic force muſt be equal to that of compreſſion, whenever the magnitude of the ſpace occupied remains the ſame. This being the caſe as long as an air bubble remains at the ſame depth, the diameter will not be altered; but as the ſpherule approaches the ſurface of the fluid, the preſſure will be diminiſhed, and conſequently the capacity increaſed in the ſame proportion: that is, the ſphere's magnitude will be in an inverſe proportion of the diſtance of its centre from the ſurface of the fluid (the preſſure of the atmoſphere not being at preſent conſidered), and conſequently the ſphere's diameter will be in an inverſe ſubtriplicate ratio of the ſame diſtance. If therefore at any given

* Robins' Gunnery.

diſtance from the ſurface a, the diameter is b, let d be the diameter at any other diſtance x, then by the rule we have $b : d :: x^{\frac{1}{3}} : a^{\frac{1}{3}}$, or $d = \frac{b\, a^{\frac{1}{3}}}{x^{\frac{1}{3}}}$.

It has already appeared, that the increaſe of the ſphere's diameter will cauſe a continual acceleration of the aſcent, and this being the real manner in which aerial ſpheres aſcend in fluids, ſhould next be conſidered.

XIV.

Fig. XXXI. Let FCE repreſent the ſurface of any fluid, B a ſpherule of air or elaſtic vapour beginning to aſcend in it, in the vertical line BC: produce BC to A, ſo that CA may be the altitude of a column of the fluid, the weight of which is equal to that of the atmoſphere preſſing againſt a given ſurface; then the velocities of the aſcending ſphere at any given points O, Q, will be in an inverſe ratio of the ſixth root of the diſtances OA, QA, when the ſphere's ſpecific gravity is diminiſhed ſine limite.

Let the ſphere's diameter when at $B = b$, and ſuppoſe the ſphere to have aſcended to the point O; and let $AO = x$, $AB = a$, then will the diameter of the ‡ ſphere at $O = \frac{b \times a^{\frac{1}{3}}}{x^{\frac{1}{3}}}$: alſo the quantity of contained air or vapour being invariable, the ſphere's ſpecific gravity will be inverſely as its magnitude; wherefore if the ſpecific gravity at B be $= m$, at O it will become $= m \times \frac{x}{a}$.

‡ Supra.

Let z be the ſpace through which a body muſt fall from reſt by the accleration of gravity, ſo that it may acquire the velocity of the ſphere in O; and ſince the force which acclerates an aſcending ſphere, the diameter of which $= d$, and its ſpecific gravity $= n$, is $\frac{1}{n} - 1 - \frac{3z}{4nd}$, ſubſtituting ‖ $\frac{ba^{\frac{1}{3}}}{x^{\frac{1}{3}}}$ for d, and $\frac{mx}{a}$ for n, ‖ Prop. XIII. ad finem. the † force which accelerates the ſphere at O will be † Sect. V. Prop. XII. $\frac{a}{mx} - 1 - \frac{3za^{\frac{2}{3}}}{4bmx^{\frac{2}{3}}}$; and ſince the fluxion of the ſpace $= -\dot{x}$, we have * $\overline{\frac{a}{xm} - 1 - \frac{3za^{\frac{2}{3}}}{4bmx^{\frac{2}{3}}}} \times \dot{x} = -\dot{z}$, * Sect. III. Prop. V. Cor. 4. & p. 56. and by reduction $\frac{4ba\dot{x}}{x^{\frac{1}{3}}} - 4mbx^{\frac{2}{3}}\dot{x} - 3za^{\frac{2}{3}}\dot{x} = -4mbx^{\frac{2}{3}}\dot{z}$, or becauſe m is ſuppoſed to be diminiſhed ſine limite by the problem $\frac{4ba}{x^{\frac{1}{3}}} = 3za^{\frac{2}{3}}$, and $z = \frac{4ba^{\frac{1}{3}}}{3x^{\frac{1}{3}}}$; and if $l = 193$ inches, the velocity of the ſphere at $O = \sqrt{\frac{16bla^{\frac{1}{3}}}{3x^{\frac{1}{3}}}} = \frac{1}{x^{\frac{1}{6}}} \times \sqrt{\frac{16bla^{\frac{1}{3}}}{3}}$; and $\sqrt{\frac{16bla^{\frac{1}{3}}}{3}}$ being conſtant, it follows, that the velocity will be inverſely as the ſixth root of x, or of the ſphere's diſtance from the point A.

When the ſpecific gravity of the ſpherule is evaneſcent, and the fluid wherein it aſcends is not preſſed by the atmoſpheres, the velocities will be in an inverſe ratio of the Fig. XXXI. ſixth root of the diſtances from the fluid's ſurface: for in this caſe, the point A will coincide with the fluid's ſurface at C.

If it be required to aſſign the velocity at any given point O, ſuppoſing the ſpecific gravity of the ſphere too conſiderable to be neglected, the problem will be more difficult: for in this caſe the relation of z and x muſt be aſſigned from the ‖equation $\frac{a\dot{x}}{mx} - \dot{x} - \frac{3za^{\frac{2}{3}}\dot{x}}{4mx^{\frac{2}{3}}b} = -\dot{z}$, or $\dot{x} - \frac{a\dot{x}}{mx} = \dot{z} - \frac{3za^{\frac{2}{3}}\dot{x}}{4mx^{\frac{2}{3}}b}$. In order to ſeparate the unknown quantities, let y be aſſumed of ſuch a magnitude that the fluent of $\dot{z}y - \frac{3za^{\frac{2}{3}}\dot{x}y}{4mx^{\frac{2}{3}}b}$ ſhall be zy, and let the whole equation be multiplied into y, the value of which is to be afterwards determined: this will give $\dot{x}y - \frac{ay\dot{x}}{mx} = \dot{z}y - \frac{3zya^{\frac{2}{3}}\dot{x}}{4mx^{\frac{2}{3}}b} = \dot{z}y + z\dot{y}$ (becauſe the fluent of $\dot{z}y - \frac{3za^{\frac{2}{3}}\dot{x}y}{4mx^{\frac{2}{3}}b}$ is by the ſuppoſition $= zy$). Subtracting $\dot{z}y$ from each ſide, and dividing by z, we have $\dot{y} = -\frac{3ya^{\frac{2}{3}}\dot{x}}{4mx^{\frac{2}{3}}b}$ and $\frac{\dot{y}}{y} = -\frac{3a^{\frac{2}{3}}\dot{x}}{4mx^{\frac{2}{3}}b}$, or log. $y = -\frac{9a^{\frac{2}{3}}x^{\frac{1}{3}}}{4mb}$; and if $e = 2.7182$,&c. it will follow, that $y = e^{-\frac{9a^{\frac{2}{3}}x^{\frac{1}{3}}}{4mb}}$ and conſequently $\dot{x}y - \frac{a\dot{x}y}{mx} = \dot{x}e^{-\frac{9a^{\frac{2}{3}}x^{\frac{1}{3}}}{4mb}} \times \overline{1 - \frac{a}{mx}}$ = the fluxion of zy; it appears therefore that the quantity ſought or $z = \frac{\text{the fluent of } \overline{\dot{x}e^{-\frac{9a^{\frac{2}{3}}x^{\frac{1}{3}}}{4mb}} \times \overline{1 - \frac{a}{mx}}}}{e^{-\frac{9a^{\frac{2}{3}}x^{\frac{1}{3}}}{4mb}}}$.

‖ Supra.

† Vid. Waring's Meditat. Analyt. p. 123.

XV.

XV.

Every thing remaining as in the laſt propoſition, let it be required to determine the time in which the ſphere of air or vapour, beginning its aſcent from B, deſcribes any given ſpace BO.

Since the fluxion of the ſpace $= -\dot{x}$, and the ſphere's velocity * at O $= \sqrt{\frac{16lba^{\frac{1}{3}}}{3x^{\frac{1}{3}}}}$, it follows, if the time required be put $= T$, that ‡ $\dot{T} = \frac{-\dot{x}\sqrt{3x^{\frac{1}{3}}}}{\sqrt{16lba^{\frac{1}{3}}}}$, and $T = -\frac{\sqrt{3}\times 6}{7}\times\frac{x^{\frac{7}{6}}}{\sqrt{16lba^{\frac{1}{3}}}} = -\sqrt{\frac{108x^{\frac{7}{3}}}{49\times 16lba^{\frac{1}{3}}}} = -\sqrt{\frac{27x^{\frac{7}{3}}}{196lba^{\frac{1}{3}}}}$; but when $x = a$, by the problem $T = 0$, the entire fluent will therefore be $T = \sqrt{\frac{27a^2}{196lb}} - \sqrt{\frac{27x^{\frac{7}{3}}}{196lba^{\frac{1}{3}}}}$ ſeconds.

* Prop. XIV.

‡ Sect. III. Prop. III.

Cor. 1. When $x = 0$, the ſphere will have aſcended through the whole ſpace to the point A, and in this caſe, the whole time of aſcent $= \sqrt{\frac{27a^2}{196lb}}$. Thus ſuppoſe a veſſel of 12 inches in height to be filled with water, and let an air bubble of $\frac{1}{100}$ part of an inch in diameter begin its aſcent from the bottom of the veſſel. Here

Fig. XXXI.

Here (the preſſure of the atmoſphere upon the water's ſurface being removed by an air pump,) to find the time of the ſphere's aſcent, we have $a = 12$, $b = \frac{1}{100}$, $l = 193$, and the time required $= \sqrt{\frac{27 \times 144}{196 \times 193 \times \frac{1}{100}}} = 3.2$ ſeconds.

Cor. 2. If two equal ſpherules of air begin to aſcend in a fluid of a given ſpecific gravity from reſt at different depths, the times of aſcent to the ſurface will be in a direct ratio of thoſe depths or of the ſpaces deſcribed by the ſpheres.

Cor. 3. If the ſphere's diameters at the firſt inſtant of motion ſhould be unequal, the times will be in a direct ratio of the depths, and an inverſe ſubduplicate ratio of the diameters.

It appeared from the preceding propoſitions, that in fluids of conſtant temperature, a ſpherical body of evaneſcent weight would aſcend uniformly; and that the velocity of * aſcent would be equal to that which a heavy body acquires in falling from reſt by the acceleration of gravity through $\frac{4}{3}$ of the ſphere's diameter, provided the diameter ſuffered no ſenſible alteration either from a variable preſſure or temperature: but when a veſſel of any liquor is placed under the receiver of an air pump, and the air either partially or almoſt wholly exhauſted, the preſſures on the air bubbles will vary conſiderably. Suppoſe BC to repreſent the depth from the ſurface through which the ſpherules aſcend from reſt; and let CA be the height of a column of the fluid equal to the preſſure of the atmoſphere: if that portion of the air be exhauſted from the receiver, which is expreſſed by the fraction $\frac{DA}{AC}$; the force of preſſure at B will be to the force at C, as $BD : CD$; which preſſures may vary in any ratio of inequality, according to the quantity of air remaining in the receiver: and this appears to be the reaſon why the air bubbles which ariſe in water, &c. when placed in vacuo, increaſe to ſo great a magnitude before they arrive at the ſurface; whereas if the preſſure of the atmoſphere is not removed from the fluid's ſurface, the diameters of the ſpherules alter not very much in aſcending through a ſmall ſpace. The rapid acceleration which is alſo obſerved in aerial ſpherules which aſcend in fluids is the

* Sect. V. Prop. XII.

Fig. XXXI.

imincluded in the exhauſted receiver of an air pump, is the immediate conſequence of their increaſing diameters; for it has been ſhewn in this caſe, that the velocity of aſcent on account of their variable ‡ diameter is in an inverſe ratio of the ſixth root of the ſphere's diſtance from the ſurface, and conſequently when it arrives at the ſurface, the velocity wou'd be greater than any that could be aſſigned, were it not for the remaining part of the air, which can never be wholly exhauſted from the receiver, and the change of the ſpherical figure cauſed by the increaſed magnitude of the cavity occupied by the elaſtic vapour.

‡ Sect. V. Prop. XIV.

There is alſo another cauſe which prevents the acceleration from being increaſed beyond a certain limit: it may be mentioned in this place: the propoſitions relating to the aſcent of aerial ſpherules are demonſtrated on a ſuppoſition, that the force whereby a body endeavours to aſcend in a fluid is equal to the difference between the weight of the ſolid and of an equal bulk of the fluid, which is true (even in a phyſical ſenſe) only while the velocity of aſcent is inconſiderable: this is the caſe when air bubbles aſcend through ſmall ſpaces in fluids expoſed to the preſſure of the atmoſphere, but when the velocity is very much increaſed, by removing the compreſſure of the atmoſphere, ſo as to bear a conſiderable proportion to that with which the fluid which impels the aſcending ſphere would ruſh into empty ſpace, if no ways impeded, the force of aſcent will be no longer equal to that aſſumed in the demonſtrations, which is always greater than the true force. This might be allowed for if neceſſary to the explication of any phænomena relating to this ſubject: but the demonſtrations which have been given are ſufficiently exact for any application to practice, where the nature of the caſe at all admits of reference to theory.

Since when the atmoſphere preſſes on the ſurfaces of fluids, the aerial ſpherules aſcend in them with velocities which are in a direct * ſubduplicate ratio of their diameters, it follows, that theſe diameters may be ſo minute that their aſcent ſhall be almoſt wholly impeded; that is, it ſhall be ſo ſlow, that a given ſpace ſhall not be deſcribed by them in any aſſignable time however great: but if by any means their diameters are increaſed, the ſpherules of air will in conſequence of this change not only become viſible to the eye, but aſcend with velocities increaſed in a ſubduplicate proportion of their diameters, if the preſſure of the fluid remains conſtant, but if the diameter be increaſed during the aſcent on account of the continual diminution of preſ-

* Sect. V. Prop. XII.

ſure, the velocity will be accelerated. Thus a veſſel of water which ſhews no appearance of theſe aſcending ſpherules in open air, if it be included under the receiver of an air pump, and the air exhauſted, the ſpherules will be ſeen to ariſe in abundance and with accelerated velocities for a conſiderable length of time. For the preſſure being removed, the ſpherules will of courſe be expanded by the elaſtic force of their contents. This proceſs extracts from the water the air contained in it.

Alſo if common water in which no ſpherules apparently aſcend, be expoſed to heat, the air bubbles will ſoon begin to ariſe in it, their diameters being enlarged, and conſequently their velocities augmented in a ſubduplicate proportion: for heat has the property of expanding all ſubſtances, eſpecially elaſtic fluids. When the degree of heat is increaſed ſo as to greatly enlarge the magnitude of the aſcending bubbles, and conſequently the rapid velocity of their aſcent, the water aſſumes the appearance of boiling; having acquired the greateſt heat which it is poſſible to communicate to it. This boiling appearance depends on the magnitude of the aſcending cavities of elaſtic vapour; but it has * been ſhewn, that a diminution of the preſſure on a fluid's ſurface, will increaſe the magnitude of the aſcending ſpheres of vapour as well as the application of fire. If therefore a portion of the air's preſſure be removed from the ſurface of a fluid, a leſs degree of heat will cauſe the appearance of boiling; it is well known, that water will boil with a leſs degree of heat when the barometer is low, than when it indicates a greater preſſure of the atmoſphere, and this decreaſe of heat correſponding to the diminution of the atmoſphere's preſſure and the heat of a boiling fluid admits of conſiderable latitude: thus if water moderately warm be placed under the receiver of an air pump, and the air be ſufficiently exhauſted, the water will be obſerved to boil. In the ſame manner newly fermented liquors which are replete with a ſpecies of vapour called fixed air, although when viewed in open air they diſcover none of the aerial bubbles or ſpherules of fixed air aſcending in them, yet when ſuch liquors are included under the receiver of an air pump and the atmoſphere removed, the air contained in the liquors will be obſerved to riſe in large and numerous bubbles, eſpecially if a degree of heat be previouſly applied; but it ſhould be obſerved, that the tenacity which this ſort of fluids generally poſſeſſes, prevents the air bubbles which ariſe from being immediately broken

* Sect. V. Prop. XIV.

and mixed with the remaining air, inſtead of which they are collected into a froth, and ſometimes riſe to a conſiderable heighth above the ſurface of the fluid.

It may be inferred from theſe principles, that air when intimately combined with the ſubſtance of water and other fluids becomes divided into parts ſo extremely minute as to eſcape the obſervation of the ſenſes, until the aerial particles are expanded into larger portions of ſpace, either by removing the preſſure of the atmoſphere; or by heat: and it appears that this minuteneſs of the particles, although of far leſs ſpecific gravity than the fluid, detains them in a quieſcent ſtate till their diameters are increaſed by one of the cauſes juſt mentioned.

It may alſo be added, that excluſive of the atmoſpheric air contained in the || ſubſtance of water, &c. ſome ſubtle ethereal elaſtic medium ſubſiſts alſo, of no ſenſible weight but capable of intumeſcence by heat: for after all the atmoſpheric air has been extracted from water, on the continued application of heat, the aſcent of bubbles will ſtill be obſerved.

|| Hawkſbee's Phyſico Mechan. Exp.

Concerning the application of the preceding propoſitions to matter of fact, it muſt be remembered, that one of the conditions on which they are demonſtrated is, that the elaſtic force of the vapour contained in the cavities of the aerial ſpherules obſerves the ſame law in regard to the ſpaces occupied by it as atmoſpheric air does; that is, the force wherewith the vapour endeavours to expand itſelf is ſuppoſed to be greater in the ſame proportion as the ſpace occupied by it is leſs. But although Mr. Robins * has demonſtrated that this law obtains in the fixed air or vapour of gunpowder, it ſeems probable, that the fixed air which riſes from ſome fermented liquors, when the preſſure on their ſurface is ſuddenly removed expands itſelf in a higher ratio than is conſiſtent with the preceding hypotheſis. This circumſtance therefore muſt be determined by accurate experiments: in the mean time it muſt be noted, that whatever law of expanſive force be obſerved in reſpect of the ſpaces occupied, the aſcent of ſpherules filled with theſe aerial fluids will not be affected, being the ſame as that which is demonſtrated in prop. XII. huj. provided the compreſſion of the fluid wherein they aſcend be invariably the ſame in all parts of their aſcent, and this compreſſion be always equal to the elaſtic force of the medium contained in the ſpherules.

* Robins' Gunnery, Prop. III.

XVI.

Vapour or ſteam conſiſts of hollow ſpherules of the fluid from which they ariſe, and their aſcent is cauſed partly by the air's gravitation, and partly by the impulſe of ſome power which acts in a direction contrary to that of the earth's gravity.

It is a known fact, that fluids expoſed to certain degrees of heat, varying with the denſity, tenacity, &c. of the fluids, are converted into a rare and elaſtic ſubſtance, called vapour or ſteam; one peculiar property of which is, that if no ways impeded it always aſcends in the atmoſphere in a direction perpendicular to the earth's ſurface.

Concerning the cauſe of this and the other phænomena obſervable in the aſcent of vapour philoſophers have adopted various opinions; none of which however ſeem to carry that perfect and ſatisfactory conviction to the mind, which the ſolution of natural phænomena from mechanical principles is ſo remarkable for. The crude and hypothetical notions which prevailed concerning this ſubject in the earlier ages of philoſophy, would naturally be exploded by men who to a diligent obſervation of nature joined thoſe helps which were to be derived from geometry. The opinion of Dr. Newentyt was, that the particles of fire adhering to thoſe of water, conſtituted what he calls moleculæ, which were ſpecifically lighter than the air, and conſequently aſcended by the ſame power which cauſes cork to riſe in water or iron in mercury. This ſolution was not ſatisfactory to Dr. Hally, who ſubſtituted in its place another more conſiſtent with the received principles of hydroſtatics. His idea was, that the aqueous particles were by the power of heat formed into hollow ſpherules, which he could ſhew, might be ſpecifically lighter than air in any aſſigned ratio. Theſe therefore muſt riſe perpendicularly upwards until they arrive at thoſe regions of the atmoſphere, the rareneſs of which is the ſame with that of the vaporous ſpherules: in this ſtate they remain until compreſſed by cold or other cauſe they become ſpecifically heavier than the ſurrounding air, and conſequently deſcend in the form of rain, dews, &c.

Although this account carries with it great appearance of probability, yet it has met with ſeveral ſtrenuous opponents, among whom is Mr. Clarke in his treatiſe on the motion of fluids; Mr. Rowning alſo uſes arguments againſt it, and after them Dr. Hamilton in a tract writ expreſsly on the ſubject, undertakes to ſhew, that Dr. Hally's hypotheſis is inſufficient to account for the phænomena of evaporation: his chief argument againſt Dr. Halley is this; if the aſcent of vapour were cauſed by water being converted by heat into an elaſtic ſteam conſiſting of hollow ſpherules, it would follow, that the quantity of ſteam raiſed from a given ſurface depends on the quantity of heat: whereas he ſhews, that evaporation is carried on even while the air is at the temperature of freezing, and that the quantity of vapour raiſed, depends more on the quantity of freſh air paſſing over a given ſurface of the fluid, than the temperature of the atmoſphere.

Not to enter into any conſideration, whether it be certain that the aſcent of vapour from warm liquor be analogous to the evaporation of fluids expoſed to the action of current air, or whether each phænomenon be inconſiſtent with the hypotheſis, that the particles which compoſe vapour are hollow ſpherules, it may be ſufficient for the preſent purpoſe, to apply the propoſitions which have been demonſtrated concerning the aſcent of ſmall ſpherical bodies in fluids of greater ſpecific gravity, to the illuſtration and examination of Dr. Hally's theory.

The denſity of ſteam is various according to its different degrees of heat. The ſteam which is raiſed from boiling water is 14000 times rarer than water, and ſince water is at a medium about 850 times rarer than air, it will follow, that aqueous vapour heated to the temperature of boiling water, is about $16\frac{1}{2}$ times lighter than air; but this will not give us the rareneſs of an individual particle or hollow ſpherule: in order to obtain the ſpecific gravity of which it muſt be conſidered, that if innumerable equal ſpheres be placed contiguous according to fig. XXXII. the empty ſpace, or interſtices between the ſpheres, will be ſomething leſs than $\frac{1}{2}$ of the whole ſpace, the proportion being as 47.6 : 100 nearly. To explain this poſition of the ſpheres more fully, as the figure of itſelf may be inſufficient, it is only neceſſary to ſuppoſe any given ſpace to be filled with hollow cubes equal to each other and contiguous: if a ſphere, of which the diameter is equal

to

to the ſide of one of theſe cubes, be placed within each cube; as one ſphere is to its circumſcribing cube, ſo will all the ſpheres be to all the cubes; and this proportion is that of 3.14159, &c. to 6, wherefore the hollow cubes being removed, the interſtices muſt be to the whole ſpace as 2.85841 : 6, or as 47.6 : 100 nearly: this proportion being the ſame whatever be the magnitude of the equal ſpheres: but if the ſpheres are ſo placed that the greateſt number poſſible may be included in a given ſpace; ſo that a plane paſſing through the centres *I*, *K*, *H*, *L*, &c. ſhall be diſtant from a plane paſſing through the centres *A*, *B*, *C*, *D* by a ſpace equal to the diameter of one of the ſpheres multiplied into $\sqrt{\frac{1}{2}}$ according to fig. XXXIII, or into $\sqrt{\frac{2}{3}}$ according to the figure adjacent; in either caſe the proportion of the interſtices to the whole ſpace will be ſomething more than that of 1 : 4, being in the ultimate proportion of $\surd\overline{18} - p$ to $\surd\overline{18}$, when the ſpheres' diameters are diminiſhed ſine limite, p being $= 3.14159$, &c. Now we cannot ſuppoſe that the aqueous hollow ſpherules are exactly diſpoſed according to fig. XXXII. ſo on the other hand, there is no ſufficient reaſon to imagine them ranged as cloſe as poſſible, as in fig. XXXIII, or in the fig. adjacent, in which the interſtices bear the ſame proportion to the whole ultimately: here then, if the knowledge of the exact manner in which the hollow ſpherules are diſpoſed were neceſſary in the preſent caſe, we could not proceed for want of data. But as the iſſue of this reaſoning will not be affected ‡ by any error that can probably be committed in aſſigning the ſpecific gravity of the aſcending ſpherules, we may aſſume it, merely to proceed with the argument, as equal to $\frac{1}{10.3}$ part of the denſity of the atmoſphere, which is about a mean ſpecific gravity between the two extremes.

‡ Vide pag. 175.

It may be objected, that a greater rareneſs is aſſigned to theſe ſpherules than they can poſſibly poſſeſs, if they be ſuppoſed to conſiſt of ſhells of water filled with air rarefied by the heat of the ſubjacent fluid: becauſe air ſuffers but a very ſmall rarefaction when heated from the temperature of freezing to that of boiling water, a given bulk of the air when at the * temperature of freezing, being expanded by the heat of boiling water into a ſpace greater than before in a proportion no greater than that of about 1. to

* Philoſ. Tranſ. for 1777.

to 1.414; and conſequently the combination of air, the ſpecific gravity of which is $\frac{1}{1.414}$ leſs, and water which is 850 times greater than that of common atmoſpheric air, can never form a ſpherule the ſpecific gravity of which ſhall be that which is juſt mentioned, i. e. $= \frac{1}{10.3}$. But it is evident from other experiments, that the cavities of the vaporous ſpherules are not occupied by air, becauſe when ſteam is condenſed by the application of cold, ſo as to be reduced to water, no air is produced from theſe cavities. The fluid which occupies the ſpherules of vapour muſt be therefore ſuppoſed to conſiſt of ſome elaſtic medium incomparably more rare and ſubtle than air: and conſequently a hollow ſphere of water filled with this vapour may be lighter than the air in any aſſigned ratio.

Let their leaſt ſpecific gravity, i. e. when at the temperature of boiling water, be aſſumed $\frac{1}{10.3}$, then by means of the different temperatures under that of boiling water to which they are expoſed, their ſpecific gravity will vary through all the intermediate magnitudes between $\frac{1}{10.3}$ and 1, according to the preceding determination: now it is to be remarked, that if theſe ſpherules were phyſically without weight, when compared with that of an equal bulk of air, their velocity of aſcent would be ſuch as a † heavy body acquires in deſcending from reſt by gravity through $\frac{4}{3}$ of their diameter, in which caſe their ſpecific gravity = 0; but when their ſpecific gravity = a fraction n, their diameter d, and $l = 193$ inches, their velocity of ‡ aſcent $V = \sqrt{\frac{16\,l\,d}{3}} \times \sqrt{\overline{1 - n}}$, and $d = \frac{3\,V^2}{16\,l \times \overline{1 - n}}$, V here ſignifying the number of inches through which the vapour aſcends in a ſecond.

It appears therefore that the velocity of aſcent will be uniform, if the diameters of the aqueous particles continue the ſame; but theſe diameters ſuffer alteration by being expoſed to different temperatures of heat and cold, which by increaſing or diminiſhing the elaſtic force of the medium occupying the cavities of the ſpherules, will of courſe

† Sect. V. Prop. XII.

‡ Sect. V. Prop. XII.

courſe expand or contract their dimenſions, and conſequently alter their diameters in a ſubtriplicate proportion.

Many of the phænomena obſervable in the aſcent of vapours are extremely conſiſtent with the preceding hypotheſis. If the ſubſtance of miſts and vapours which are ſeen to ariſe from lakes or damp grounds, be formed of hollow ſpherules as above deſcribed, they will aſcend according to the theory with a motion wholly uniform as long as their diameters are invariable, which is conſiſtent with obſervation; the diameters of the vaporous particles being unaltered while the air's temperature continues the ſame: alſo ſince it is the property of all elaſtic fluids to be expanded by heat, we may juſtly ſuppoſe, that the magnitude of theſe ſpherules will be enlarged by the ſame means; an immediate conſequence as deduced from the theory is, that their velocity of aſcent will be increaſed in a ſubduplicate ‡ ratio of their diameters. This alſo is conſiſtent with obſervation, for while vapours hang as it were ſuſpended in the atmoſphere, their ſpecific gravity being but little different from that of air, the heat of the ſuns rays by expanding them into larger portions of ſpace, accelerates the velocity of their aſcent, and from theſe principles it would naturally follow, that the diameter of the particles of ſteam which riſes from boiling water ſhould be larger than thoſe of ſpherules which conſtitute vapour when heated to any other temperature, and conſequently the uniform velocity of its aſcent ſhould be greater than that of ſteam heated to any inferior degree, which is alſo conſiſtent with experience.

‡ Sect. V. Prop. XII. Cor. 2.

Thus far theory and matter of fact ſeem to agree; and if every other appearance were equally deducible from the hypotheſis, we ſhould not heſitate to pronounce, that the gravitation of the atmoſphere on the ſpherules ſpecifically lighter was the only cauſe by which vapour aſcended: but a difficulty here occurs from the mathematical principles, which if they are not ſufficient for the intire explication of the phænomena, are yet ſerviceable at leaſt in ſubjecting the hypotheſis to examination.

If the diameters of the aqueous ſpherules be given, together with their ſpecific gravity in reference to that of the air wherein they aſcend, their velocity of aſcent may be determined a priori as far as it is effected by the air's gravitation, and converſely, if the velocity wherewith vapour aſcends, and the ſpecific gravity of the globules be given, their diameter may be aſcertained.

If therefore the vapour which riſes from boiling water conſiſts of hollow ſpherules, and the velocity with which it aſcends in the air be obſerved, the diameter of the ſpherules will be determined, and ſhould agree with obſervation if the theory be perfect: but it is evident, that the particles whereof ſteam conſiſts are of ſuch minuteneſs as to render it impoſſible to obſerve them ſeparate from each other, the ſubſtance of ſteam appearing, even when examined by glaſſes, as a rare and perfectly continuous maſs by no means diſtinguiſhable into parts; whereas the diameter of the particles derived from the theory, upon a ſuppoſition that the air's preſſure is the only cauſe of their aſcent, is of ſuch a magnitude as would render each ſpherule obvious to the ſenſes: for if we obſerve the aſcent of ſteam from the ſurface of boiling water, it will be found to riſe at the rate of 4 inches in a ſecond: and ſince the ſpecific gravity of a particle is about $\frac{1}{10.3}$, that of air being 1, we have for the determination of the diameter d from the theory, $n = \frac{1}{10.3}$, $l = 193$ inches, $V = 4$, and the diameter* required $= \frac{3V^2}{16l \times \overline{1-n}} = \frac{1}{58}$ part of an inch.

It is to be obſerved, that this reaſoning is not affected by any error in the determination of the ſpecific gravity of the aqueous ſpherules which riſe from boiling water; for their ſpecific gravity was aſſumed $\frac{1}{10.3} = n$, that of the air being 1, let n be increaſed, the diameter $d = \frac{3V^2}{16l \times \overline{1-n}}$ will be alſo increaſed, and if the ſpherule's ſpecific gravity be diminiſhed, the diameter of the ſpherule will decreaſe, but can never be leſs than $\frac{1}{64}$ which is its limit as when n is $= 0$, and the ſame difficulty occurs from ſuppoſing the diameter $= \frac{1}{64}$ as $\frac{1}{58}$ of an inch.

It is however certain, that the diameters of the particles of ſteam which riſe from boiling water, are far leſs

* Sect. V. Prop. XII.

than $\frac{1}{64}$ of an inch: let the diameter, for example, be ſuppoſed to $= \frac{1}{120}$ of an inch: to find the velocity of their aſcent, we have $n = \frac{1}{120}$, $l = 193$, and V the velocity required $= \sqrt{\frac{16ld}{3}} \times \sqrt{1 - n} = 2.78$, or not ſo much as 3 inches in a ſecond; but the ſteam riſes at the rate of about 4 inches in the ſame time: it would follow therefore, upon the ſuppoſition that the diameter of a particle of ſteam $= \frac{1}{120}$ part of an inch, that this additional velocity over and above that which is cauſed by the air's preſſure muſt be attributed to ſome power which acts in a direction contrary to that of the earth's gravity.

It is rather analogous to the operations of nature, than inconſiſtent with them, to ſuppoſe the exiſtence of ſome ſuch power acting in a direction contrary to that of gravity. The remarkable phænomenon of the perpendicular growth and poſition of plants and trees muſt be attributed to the conſtant agency of ſome force external to the plant or tree itſelf; for if the ground on which a vegetable is planted be inclined to the horizon at any angle whatever, the plant will however ſoon obtain a direction perpendicular to the horizon, and continue to increaſe in that direction. Another phænomenon may be here mentioned alſo: if a bar of metal be fixed in a vertical direction, and heat be by any means applied to the middle of it, this heat will be communicated to the upper part of the bar, when the temperature of the lower part is ſcarcely altered.

Many other inſtances might be urged to prove the exiſtence of ſome power acting in a direction contrary to that of the earth's gravity: neither will this hypotheſis be at all inconſiſtent with that general phyſical truth, i. e. that the weight of every body is proportional to the quantity of matter contained in it; for this will be the caſe whether bodies be acted on by gravity alone, or by two powers in contrary directions, the ſtrongeſt of which is gravity.

It muſt be however remarked, that this power which ſeems neceſſary for the elevation of ſteam, cannot be proved the only cauſe of the phænomenon: many facts and arguments tending to ſhew that the gravitation of the air is greatly concerned in producing them, although it be alone inſufficient to account for them wholly. Thus, when we ſee the ſmoke which is impregnated with the matter from which it ariſes aſcending in the open air, and deſcending when placed in a rarer medium within the receiver of an air pump, we muſt conclude that the air's preſſure was neceſſary to the aſcent, and that the groſs nature of the matter carried up with the vapour, renders its weight ſuperior to the force whereby it is impelled upwards. Alſo the phænomena of the increaſed velocity of aſcent in vapour from the increaſe of heat tends greatly to ſtrengthen this opinion, it being very conſiſtent with the theory of the aſcent of ſpherules in air above deſcribed.

It may be added to theſe obſervations, that during the aſcent of vapour into the regions of the clouds, two cauſes ſeem to operate in a manner contrary to each other. For while the increaſing rareneſs of the air cauſes an expanſion of the ſpherules, by removing † the compreſſure upon their ſurfaces, and conſequently increaſes their velocity of aſcent, the colder temperature of the higher regions by contracting the dimenſion of the ſpherical cavities diminiſhes the velocity, and according to the degrees in which theſe cauſes operate, the vapour will ariſe with a greater or leſs degree of velocity until it becomes ſtationary when theſe cauſes counterbalance each other. Alſo it is eaſy to imagine, that atmoſpheric cold may render the vapour ſo denſe as to make it preponderate againſt the force which raiſed it, ſo that it ſhall deſcend in various forms, i. e. in rains, miſts, dews, &c. according to its degree of condenſation.

† Prop. XIV.

XVII.

Let a ſpherical body conſiſting of elaſtic vapour, the expanſive force of which is inverſely proportional to the ſpace it occupies, aſcend in air of uniform denſity; and let the ſphere be of evaneſcent weight when compared with that of an equal bulk of air, if the air's denſity be diminiſhed in the proportion of n : 1, the velocity of the ſphere's aſcent will be increaſed in the proportion of $1 : n^{\frac{1}{6}}$.

For if the denſity of the air be diminiſhed in the ratio of $n : 1$, its elaſtic force as well as the force of preſſure will be diminiſhed in the ſame proportion: the ſpace occupied by the ſphere will therefore be increaſed in the proportion of $1 : n$, and conſequently the ſphere's diameter increaſed in that of $1 : n^{\frac{1}{3}}$: but ſpheres of evaneſcent weight aſcend in fluids with velocities, which are in the direct ſubduplicate * ratio of their diameters; the velocities therefore with which the ſpheres, whoſe diameters are as $1 : n^{\frac{1}{3}}$ aſcend, are in the proportion of $1 : n^{\frac{1}{6}}$.

From this propoſition may be deduced the explanation of ſeveral phænomena relating to the aſcent of vapour, that is, as far as the air's gravitation is concerned in producing them. Whatever be the original cauſe by which the aqueous ſpherules are detached from the ſubjacent fluid, it ſeems obvious, that the ſame power which contributes to promote the velocity of their aſcent, muſt alſo tend to augment the quantity of evaporation. From the theory two cauſes have been deduced whereby this effect is produced, i. e. heat, or a * diminution of the atmoſphere's gravity, both of which however operate by the ſame means, that is, by increaſing the magnitudes of the aſcending ſpherules, and in theſe particulars theory and matter of fact are intirely conſiſtent: for it is obvious to common obſervation, that an increaſe of heat augments the quantity of evaporation under a given preſſure of the atmoſphere, and experiment demonſtrates, that fluids of a given temperature being placed under the receiver of an air pump evaporate the faſter the more the air is exhauſted; ſo that when either pure water or any moiſt ſubſtance is placed under a receiver, as the air is exhauſting the water becomes converted into vapour, riſing in greater abundance according to the rareneſs of the air within the receiver. This vapour by its elaſticity ſupplies in part the air exhauſted, and is a very great impediment to the due operation and effect of the air pump, as well as a cauſe of un-

* Sect. V. Prop. XII.

* Page 168.

uncertainty in eſtimating the degrees of exhauſtion which that inſtrument is capable of producing. In order therefore to exhauſt a receiver as perfectly as poſſible, no moiſture of any kind nor any ſolid body which may have imbibed it ſhould communicate with the receiver. This has been fully proved by the author of an ingenious and intelligent ſeries of experiments lately publiſhed. ‡

It is manifeſt that the denſity of fluids will not obſtruct their evaporation ſo much as their tenacity: a hollow ſpherule of mercury will riſe equally faſt in air with one of water of the ſame diameter, provided the ſpherical ſhell of mercury be about 14 times thiner than that of water, both being ſuppoſed to be filled with an elaſtic fluid in a phyſical ſenſe void of weight; and the very ſubtle and diviſible nature of material ſubſtance in general, and eſpecially of this fluid, renders any degree of tenuity in the ſhells of the mercurial ſpherules eaſily imaginable. Moreover, the parts of this fluid being intirely free from coheſion or tenacity, it will readily be converted into vapour by a moderate degree of heat, and in the aſcent of theſe vapours the ſame laws will be obſerved as in the aſcent of thoſe which ariſe from water. Thus taking off the preſſure of the atmoſphere from the ſurface of mercury will facilitate and accelerate the aſcent of vapour from it; in whatever degree the air be rarefied this aſcent would ſtill continue were the † mercurial ſhells abſolutely without weight: and it is poſſible, that there is no vacuum to be made by art, but what will contain air ſufficient to elevate the attenuated ſpherules: it is certain, that they occupy the ſpace between the ſurface of the mercury and the upper extremity of a barometer,* the altitude of the mercury being ſenſibly affected thereby; but whether this aſcent of mercurial vapour ſhould be attributed to the very ſmall quantity of air, which notwithſtanding the utmoſt care uſed in conſtructing the inſtrument may remain within the ſubſtance of the mercury, and at length emerge from its ſurface, or to that additional power over and above the air's gravitation which has appeared neceſſary to account for the phænomena of the aſcent of || vapour, future experiments muſt determine.

To examine this ſubject a little more particularly: the ſmoke which ariſes from ignited bodies, it is evident muſt aſcend by the ſame cauſes which elevate vapour from the pure fluids, and this being the caſe, it might be expected that the laws obſerved in the aſcent of them all would be preciſely the ſame: but although vapour from water and mer-

‡ Philoſ. Tranſact. for 1777, p. 614.

† Prop. XII. Cor. 1.

* Philoſ. Tranſ. for 1777. p. 671.

|| Sect. V. Prop. XVI.

mercury riſe the more copiouſly and with the greater velocity, according to the rareneſs of the air wherein they aſcend, ſmoke from ignited bodies deſcends in air which is rarefied to any conſiderable degree: and this phænomenon is by no means inconſiſtent with the theory; for the globules which form ſmoke are always of an unctuous nature, and conſequently require far greater degree of heat to expand them ſo as to become capable aſcending, than that by which the vapour of mercury or water begins to riſe: but when once the globules of ſmoke have aſcended from the ignited body, being deprived of their heat, they ſoon become hard in cooling, and are incapable of any expanſion from the elaſticity of the contained medium, notwithſtanding the external air ſhould be rarefied in any degree however great. Now it *appeared, that the rarefaction of the air in which the vapour of water or mercury aſcended, would cauſe an acceleration of the vapour's velocity from the increaſe of the diameter of the globule's which from their extreme fluidity expand themſelves into larger portions of ſpace as the external preſſure of the air is diminiſhed; but in ſmoke the tenacious quality of the hollow globules when cool prevents ſuch expanſion, and conſequently when the air is ſo rarefied that the weight of a globule exceeds that of an equal bulk of air (together with whatever additional force over and above the air's preſſure contributed to its aſcent) it will deſcend.

* Prop. XVII.

The action of any other power, except that of the air's preſſure has not been very particularly conſidered as it is not here intended to ſet forth a compleat theory, but to apply the mathematical principles as far as they will allow us to the explication of the phænomena. What has been obſerved in relation to ſmoke is applicable in ſome degree to the vapour which ariſes from heterogeneous fluids, and from ſolids impregnated by them. Here the globules of vapour which ariſe from the ſame ſubſtance will differ in quality as well as their velocity of aſcent; and while the purer and moſt fluid parts continue to aſcend with additional velocity as the air is ‡ rendered more rare, the groſſer vapour, the tenacity of which admits not of an expanſion of the globule's diameters like the former, will either riſe with a leſs degree of velocity, continue ſtationary, or deſcend, as the tenacity operates in a greater or leſs degree while the air is rarefied.

‡ Prop. XVII.

SECT.

SECT. VI.

CONCERNING THE COMMUNICATION OF MOTION TO BODIES WHICH REVOLVE ROUND A FIXED AXIS.

THE principal properties of rectilinear motion, both accelerated and retarded, may be inferred from the preceding propoſitions; in demonſtrating theſe, the impelling force, as well as that of reſiſtance, has always been ſuppoſed to be impreſſed in the direction of a ſtraight line paſſing through the centre of gravity of the moving body, and in this caſe every particle of the body muſt partake of the ſame degree of velocity, being equal to that with which the common centre of gravity moves.

But it frequently happens that a body or ſyſtem of bodies is ſo conſtituted, that when any force is impreſſed upon it, no motion can be produced, except round a fixed axis; ſo that the velocity of the particles which compoſe the ſyſtem will be greater or leſs, according as theſe particles are further from the common axis or nearer to it. Theſe circumſtances ſhould be attended to, in order to aſcertain the motion of revolving bodies, the preceding principles of acceleration being not wholly of themſelves ſufficient for that purpoſe, without taking into conſideration the particular nature of rotatory motions.

In order to demonſtrate the laws obſerved by bodies which revolve round a fixed axis, two things are principally to be attended to, i.e. the moving force by which the revolving motion is generated, and the inertia of the parts whereof the ſyſtem is compoſed; the moving force exerted on any given particle in the ſyſtem as well as its inertia depends on its diſtance from the axis of motion, every thing elſe being the ſame, and if both theſe be aſcertained, the abſolute acceleration of the particle will be determined, and conſequently the abſolute velocity generated in it in a given time.

The methods therefore of determining theſe forces in any given circumſtances ſhould next be deſcribed.

Let AFGH repreſent the circumference of a wheel which turns in its own plane round an horizontal axis, paſſing through

S its centre, and let a weight P, fixed at the extremity of a line A P, communicate Fig. XXXIV. motion to the wheel. Moreover, let the whole weight of the wheel be Q; and ſuppoſe this weight to be collected uniformly into the circumference A F G H, then during the deſcent of the weight P, each point of the circumference muſt move with a velocity equal to that with which P deſcends, and conſequently ſince the moving force is the weight P, and the maſs moved P + Q, the force which accelerates P in its deſcent, will be that part of the accelerating force of gravity which is || expreſſed by the fraction $\frac{P}{P+Q}$. || Sect. I. Prop. IX. The velocity therefore which is generated in P in any given time is found from the rules before * demonſtrated: thus, ſuppoſe * Sect. III, Prop. I. Q to be equal to P, then will $\frac{P}{P+Q} = \frac{1}{2}$, and the weight P will be accelerated by a force which is to that of gravity as 1 : 2. and ſince gravity generates in bodies which deſcend near the earth's ſurface one ſecond of time a velocity of $32\frac{2}{12}$ feet in a ſecond, it follows, that the weight P will in the ſame time have ‡ acquired in its deſcent a ‡ II. Law of Motion, or Sect. II. Prop. I.

velocity of $16\frac{1}{12}$ feet in a ſecond only. This caſe is mentioned rather particularly, as the more complicated properties of revolving bodies demonſtrated in the ſubſequent propoſitions will be referred to it.

The parts of the weight Q which are uniformly diſpoſed over the circumference AFGH, balance each other round the common centre of gravity S, their weight therefore is of no effect in accelerating or retarding the deſcent of P; and this will be the caſe whenever the axis of motion paſſes through the common centre of gravity: but in order to render the properties of rotatory motions more obvious, it will be convenient to diſpoſe the parts of the revolving ſyſtem, ſo that the axis of motion ſhall not neceſſarily paſs through the common centre of gravity: thus, referring to the preceding caſe, inſtead of ſuppoſing the weight Q to be uniformly diſpoſed over the circumference AFGH, let it be collected into any point Q. Here it is manifeſt, that if the maſs Q be acted on by gravity, the force which communicates motion to the ſyſtem round S will be variable, it being the greateſt when SQ is horizontal, and gradually diminiſhing till Q has deſcended to its loweſt point.

But as we ſhould begin with the moſt ſimple caſes, the moving force muſt be conſtant. This will be effected by ſuppoſing the maſs which is collected in Q to be deſtitute of weight, and to poſſeſs inertia only; it follows therefore that during the revolution of Q round S as an axis, the moving force will be conſtantly equal to P, and the maſs moved $= P + Q$, and conſequently the force which accelerates the deſcending weight or any point in the circumference will be that part of gravity which is expreſſed by the *fraction $\frac{P}{P + Q}$ the ſame as before.

In theſe caſes the force which communicates motion to the ſyſtem, has been ſuppoſed a weight or body acted on by the earth's gravity, and conſequently conſtitutes a part of the maſs moved, at the ſame time it acts as a moving force: but motion may be communicated by a force which ſhall add nothing to the inertia of the matter moved, and it will be convenient in many demonſtrations to aſſume the moving force of this kind: the inertia of the moving force P therefore in the ſubſequent propoſitions

* Sect. I. Prop. IX.

will not be taken into account, unleſs expreſſly mentioned. Thus, if any number of bodies without gravity, being collected into the points F, H, Q, are cauſed to revolve round the axis S, by a moving force P, the force which accelerates theſe bodies in their ‡ revolution will be $\frac{P}{F + H + Q}$; provided the bodies F, H, Q be diſpoſed at a diſtance from the axis of motion equal to the radius of the circle A F G H, at the circumference of which the moving force P is applied.

Fig. XXXIV.

‡ Sect. I. Prop. IX.

In the preceding example F, H, Q, &c. have been ſuppoſed to move with the ſame velocity: but when bodies revolve at unequal diſtances from the axis, their velocities of motion being different, other rules will be neceſſary to determine the force whereby any given point of the ſyſtem is accelerated; theſe are contained in the following propoſitions. In demonſtrating the properties of rotatory motion, the revolving ſyſtem will be ſuppoſed to conſiſt of one or more of the bodies A, B, C: the magnitude of theſe is aſſumed as evaneſcent, becauſe were the contrary ſuppoſition adopted, the particles in each body would be impelled by different moving forces, and exert different degrees of inertia in oppoſition to the communication of motion. But the force which impels each individual particle, and the effects of its inertia in given circumſtances muſt be known before the acceleration of the whole ſyſtem can be determined.

The bodies A, B, C, which may be termed, according to the ideas juſt deſcribed, material points, are imagined to be connected together by ſome perfectly rigid ſubſtance, ſo as to always poſſeſs the ſame ſituation in reſpect of each other, and conſequently no motion can be produced in any of them, except all revolve at the ſame time round the common axis of motion.

All the points in this imaginary ſubſtance by which the parts of the ſyſtem are connected together, partake of the ſame angular motion deſcribing circles round the common axis S; a force P therefore being applied to any point in the plane of its motion, and in the direction of any line in that plane which paſſes not through the axis, will communicate an equal angular motion to the whole. Thus, let B repreſent a material point Fig. XXXV.

point moveable about an axis of motion paſſing through S, with the centre S, and diſtance SD, deſcribe a circle DGH: now, if B be connected with every point in the area of this circle, which is an inflexible ſubſtance, no force can be applied to move the circle, but what muſt communicate the ſame angular motion to B. Let the force be applied at the point D; it is manifeſt, that in order render its effects conſtant, the inclination of its direction to SD muſt be always the ſame, and in a given plane: and the moſt obvious method of effecting this, either in conſidering the ſubject theoretically, or in the practical illuſtration of it, is by applying a thin and flexible line GHDP round the circumference of the circle DGH, and ſtretching this line by a given moving force P. Here it is plain, that in whatever part of the circumference DGH, the point D is ſituated, the effects of the force P will be the ſame, as if it were directly applied to D in the direction of the plane of motion and perpendicular to SD, and the point B will revolve with the ſame abſolute and angular velocity in both caſes.

Theſe conſiderations being premiſed, in order to prevent unneceſſary repetitions,

tions, let A, B, C be a ſyſtem of bodies of evaneſcent magnitude and without gravity moveable about an axis of motion which paſſes through S, it is to be noted, that the imaginary ſubſtance by which parts of the ſyſtem A, B, C are connected, muſt contribute nothing, either by its weight or inertia, to accelerate or retard the motion of the material points A, B, C, which are cauſed to revolve by the action of the given and conſtant force P applied at the diſtance from the axis S D. Fig. XXXVI.

The abſolute force of P to move D, or any point in the circumference, will be P, but the communication of motion to this point D is reſiſted by the inertia of the bodies A, B, C, which being moved with different velocities and acted on by different moving forces, their inertia will not be eſtimated by their * quantities of matter only, according to the laws obſerved in rectilinear motion: the force which accelerates D therefore cannot be obtained by dividing P by A + B + C; but if an equivalent maſs, or a quantity of matter can be aſſigned, which being collected into any points of the circumference a, b, c will cauſe an inertia or reſiſtance to the motion of D, equal to that exerted

* Sect. I, Prop. IX.

erted by the particles A, B, C, when revolving at their reſpective diſtances, the force which accelerates the circumference or any point in it D will be determined. Thus, let the maſs Q when collected into a, be ſuch as will be equivalent in its inertia to A when revolving at the diſtance SA; alſo, let R be the maſs collected into b, which is equivalent to B when revolving at the diſtance SB; and let T be the maſs collected in c be equivalent to C when revolving at the diſtance SC; then will the maſs moved by the force P be $Q + R + T$, and the force which accelerates the circumference $= \frac{P}{Q + R + T}$*, being equal to that by which the circumference or any point in it is accelerated, when the ſyſtem conſiſts of A, B and C, revolving at the reſpective diſtances from the axis of motion SA, SB, SC.

N. B. In the ſubſequent propoſitions the moving force *P* is ſuppoſed to be applied to any point *D*, in the direction of a plane which is perpendicular to the axis of motion, and at right angles to *SD*.

I.

Fig. XXXV.

Let a body collected into the point B, revolve round an axis of motion paſſing through S by the action of a force P; applied at the diſtance S D from the axis; it is required to aſſign what quantity of matter muſt be collected in the point D, ſo that D may be accelerated in the ſame manner as when B revolved at the diſtance S B, and conſequently that the angular velocity of B and D round S may be the ſame in both caſes.

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* Sect. I. Prop. IX.

When any two bodies are put in motion by two conſtant forces acting for the ſame time, the quantities of matter moved are in a direct || ratio of the moving forces and an inverſe ratio of the velocities generated; that is, if $\frac{M}{m}$ expreſſes the ratio of the moving forces, $\frac{Q}{q}$ that of the quantities of matter, and $\frac{V}{v}$ of the velocities generated, the relations of theſe quantities is defined by the equation $\frac{Q}{q} = \frac{M}{m} \times \frac{v}{V}$.

|| Sect. III. Prop. VII.

To apply this it muſt be obſerved, that although the abſolute force of the weight P acting upon the point D, remains conſtantly the ſame, yet its effects upon bodies placed at different diſtances from the axis of motion, are in the inverſe proportion of thoſe diſtances; therefore the moving forces exerted by P on the points B and D, will be in the proportion of SD to SB: alſo by the problem the angular motion of D and B are equal, and conſequently the velocity of B is to the velocity of D, as $SB : SD$; and ſince the quantities of matter ‡ in B and D are in the direct proportion of the moving forces, or of $SD : SB$, and an inverſe proportion of the velocities generated, or of $SB : SD$, we ſhall have the quantity of matter in B to that contained in D, as $SD^2 : SB^2$, and conſequently the weight ſought $= B \times \frac{SB^2}{SD^2}$.

‡ Supra.

Or thus, let $x =$ the quantity of matter required to be collected in D, M the moving force which acts on B, m that which acts on D, V the velocity of B, v that of D,

‡ Sect. III. Prop. VII. then ‡ will $\frac{B}{x} = \frac{M}{m} \times \frac{v}{V}$, but $\frac{M}{m} = \frac{SD}{SB}$ by the property of the lever: moreover $\frac{v}{V} = \frac{SD}{SB}$ by the nature of angular motion, wherefore $\frac{B}{x} = \frac{SD^2}{SB^2}$, and $x = B \times \frac{SB^2}{SD^2}$.

Cor. 1. Whenever therefore a body B revolves round an axis by the action of a conſtant force P, applied at a given diſtance SD from the axis; in order to find the force which accelerates D, the maſs B may be ſuppoſed to be removed, and inſtead of it an equivalent weight $B \times \frac{SB^2}{SD^2}$, collected in the point D to which the force is applied.

† Sect. I. Prop. IX. Cor. 2. Since the † moving force is P, and the maſs moved $\frac{B \times SB^2}{SD^2}$, (the inertia of P not being conſidered,)

‖ Page 185. the force which accelerates D will be ‖ that part of the acceleration of gravity which is expreſſed by the fraction

$$\frac{P}{B \times \frac{SB^2}{SD^2}} = \frac{P \times SD^2}{B \times SB^2}.$$

Fig. XXXVI. Cor. 3. Let any number of bodies A, B, C, &c. be put in motion round a fixed axis, paſſing through S by a conſtant force P applied at D, the point D will be accelerated in the ſame manner, and conſequently the whole ſyſtem will have the ſame angular velocity, if inſtead of A, B, C, &c. placed at the diſtances SA, SB, SC, we ſubſtitute the bodies $\frac{A \times SA^2}{SD^2} + \frac{B \times SB^2}{SD^2} + \frac{C \times SC^2}{SD^2}$, theſe being collected into the points a, b and c reſpectively: the moving force in this caſe is P;

* Sect. VI Prop. I. and p. 185. the maſs * moved $= \frac{A \times SA^2}{SD^2} + \frac{B \times SB^2}{SD^2} + \frac{C \times SC^2}{SD^2}$; and conſequently the force which accelerates D is that part of the accelerating force of gravity which is expreſſed by

the

the fraction $\dfrac{P}{A \times \dfrac{SA^2}{SD^2} + B \times \dfrac{SB^2}{SD^2} + C \times \dfrac{SC^2}{SD^2}} = \dfrac{SD^2 \times P}{A \times SA^2 + B \times SB^2 + C \times SC^2}$.

Cor. 4. The velocity|| of the point *D* is uniformly accelerated, becauſe the force above determined is invariable: it follows alſo, that the angular velocity of the ſyſtem is uniformly accelerated, becauſe the abſolute velocity of any point at a given diſtance from the axis of motion is as the angular velocity of that point, and conſequently of the whole ſyſtem. || Sect. I. Prop. IV.

Cor. 5. It is manifeſt that it is of no conſequence whether the bodies *A*, *B*, *C*, &c. revolve in the ſame or in different planes, if their diſtances from the axis *S A*, *S B*, *S C* are the ſame; theſe diſtances being eſtimated by lines drawn through *A*, *B*, and *C* perpendicular to the common axis of motion; if therefore they ſhould be ſituated in various planes, they may be referred to any one given plane perpendicular to the axis.

Cor. 6. It is obvious alſo that changing the poſition of the bodies *A*, *B*, *C* in the ſame plane will not affect the force which accelerates the ſyſtem, provided their reſpective diſtances from the axis of motion alter not: thus, with the centre *S*, and diſtances *SB*, *SC*, let the arcs of circles be deſcribed; if *B* is transferred to *b*, or *C* to *c*, the moving force which acts on theſe bodies reſpectively will not be altered, and conſequently the maſſes moved being likewiſe conſtant, the accelerating force will be the ſame. Fig. XXXVI.

Cor. 7. Alſo the propoſitions concerning rotation will be equally true, whatever be the force by which the angular motion is generated, provided it be conſtant; or if variable ſhould its action be conſidered for an evaneſcent particle of time only.

Cor. 8. In theſe propoſitions the angular motion is ſuppoſed to be communicated by the action of a force applied at a ſingle point only, at an invariable diſtance from the axis. Thus, ſuppoſe a revolving ſyſtem to conſiſt of the bodies *A*, *B*, *C* of evaneſcent magnitudes and equal to each other in quantity. Let a weight *P* equal to *A*, *B*, or *C* be applied by means of a line going round a wheel *DH*, ſo as to communicate motion to the ſyſtem during its deſcent. Moreover, let $SD = 1$, $SA = 2$, $SB = 3$, and $SC = 4$, and let it be required to aſſign the force which accelerates the deſcent of the weight *P*. Fig. XXXVI.

Here

Here referring to the rule, we have the force required $= \frac{SD^2 \times P}{A \times AS^2 + B \times SB^2 + C \times SC^2} = \frac{1}{4+9+16} = \frac{1}{29}$, which is the true accelerating force when the moving force is void of inertia; but in the preſent caſe, the weight P acts not only as a moving force, but as a part of the maſs moved, and as P moves with the ſame velocity as the point D, the effects of its inertia will be eſtimated by ſuppoſing a maſs of matter $= P$ to be collected into the point D: here then the moving force being P, and the maſs moved $P \times \frac{SD^2}{SD^2} + A \times \frac{AS^2}{SD^2} + B \times \frac{BS^2}{SD^2} + C \times \frac{CS^2}{SD^2}$, the force* which accelerates the deſcent of P will be $\frac{SD^2 \times P}{P \times SD^2 + A \times AS^2 + B \times SB^2 + C \times SC^2} = \frac{1}{30}$, the line and parts of the wheel not being taken into account. It follows therefore, that the weight P will deſcend from reſt by the force of this acceleration through $\frac{16\frac{1}{12}}{30}$ parts of a foot in a ſecond; and in other different times of deſcent the ſpaces † will be varied in a direct duplicate ratio of the times, becauſe the accelerating force is conſtant.

Moreover, the velocity acquired by the deſcending weight P in a given time, or through a given ſpace, will be determined by the rule demonſtrated in ſect. III. prop. V. Thus putting $\frac{P \times SD^2}{P \times SD^2 + A \times AS^2 + B \times BS^2 + C \times CS^2} = F$, $l = 193$ inches, $s =$ the ſpace deſcribed by P from reſt, the velocity acquired by P will be $\sqrt{4lFs}$ inches in a ſecond, and in the ſame manner having given any two of the four quantities, i. e. the accelerating force, the ſpace deſcribed from reſt, the time of deſcription, and the velocity acquired, the other two may be determined as in ſect. III.

For the further illuſtration of this ſubject, the following property of uniformly accelerated motion may be here inſerted.

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* Sect. I. Prop. IX.

† Sect. III. Prop. IV.

II.

If any two bodies be acted on by moving forces which are conſtant, and which are in an inverſe proportion of the ſpaces deſcribed from reſt, the quantities of matter contained in the bodies will be in an inverſe duplicate ratio of the velocities generated; that is, if $\frac{Q}{q}$ repreſents the ratio of any two maſſes of matter, $\frac{M}{m}$ the ratio of the moving forces by which they are impelled, $\frac{S}{s}$ that of the ſpaces deſcribed from reſt, and $\frac{V}{v}$ the ratio of the velocities generated, the propoſition aſſerts, if $\frac{M}{m} = \frac{s}{S}$, that $\frac{Q}{q} = \frac{v^2}{V^2}$.

The duplicate ratio of the velocities is equal to the ſum of the ratios of the ſpaces deſcribed and of the || accelerating forces, or $\frac{V^2}{v^2} = \frac{F}{f} \times \frac{S}{s}$, but * $\frac{F}{f} = \frac{M}{m} \times \frac{q}{Q}$, we have therefore $\frac{V^2}{v^2} = \frac{M}{m} \times \frac{S}{s} \times \frac{q}{Q}$, and when as in the problem $\frac{M}{m} = \frac{s}{S}$, or $\frac{M}{m} \times \frac{S}{s} = 1$, the equation which defines

the

|| Sect. III. Prop. V.

* Sect. I. Prop. IX.

the velocities and quantities of matter will be $\frac{V^2}{v^2} = \frac{q}{Q}$, that is, the quantities of matter are in an inverſe duplicate ratio of the velocities generated.

This concluſion immediately applies to the preceding propoſition, in which it is required to aſſign the ratio of two quantities of matter under the following conditions, i. e. 1. They are to revolve round a common axis S with the ſame angular velocity, the abſolute velocities therefore muſt be in the direct proportion of their diſtances from the axis, or of the ſpaces deſcribed. 2. While a conſtant force P is applied to any given point D, the effects of theſe forces to impel bodies at different diſtances from the axis will be in an inverſe ratio of thoſe diſtances, we have therefore the ſpaces deſcribed in an inverſe ratio of the forces which impel the two bodies, and conſequently, by the propoſition juſt demonſtrated, the quantities of matter in an inverſe duplicate proportion of the velocities generated, or of the diſtances from the axis.

Cor. 1. Let it be required to aſſign what quantity of matter muſt be collected into any point E, ſo that the ſyſtem in fig. xxxv. ſhall revolve with the ſame angular velocity as when B revolved at the diſtance SB, every thing elſe remaining, by the rule we have the weight ſought to be collected in $E = \frac{B \times SB^2}{SE^2}$.

III.

In a ſyſtem of bodies which revolve round a fixed axis with an uniformly accelerated motion, the forces which accelerate particles at different diſtances from the axis will be in a direct ratio of thoſe diſtances.

By the nature of angular motion, the velocities of the different points in any revolving ſyſtem will be in a direct ratio of their diſtances from the axis of motion: but whenever bodies are ‡ uniformly accelerated, the velocities generated

‡ Sect. III. Prop. I.

generated in a given time will be directly as the accelerating forces; and conſequently the forces which accelerate the different points in a ſyſtem of revolving bodies, muſt be in a direct ratio of thoſe diſtances from the axis.

Cor. 1. It followed from prop. 1. cor. 3. that the force which accelerated the point D was $= \dfrac{P \times SD^2}{A \times SA^2 + B \times SB^2 + C \times SC^2}$, and by † this propoſition, the force which accelerates D is to that by which B is accelerated in the proportion of $SD : SB$: it follows therefore, that the force which accelerates B $= \dfrac{P \times SB \times SD}{A \times SA^2 + B \times SB^2 + C \times SC^2}$; in like manner the force which accelerates any other point C $= \dfrac{P \times SC \times SD}{A \times SA^2 + B \times SB^2 + C \times SC^2}$, and conſequently the the ſum of the forces which accelerate all the particles A, B, C will be $\dfrac{\overline{AS + SB + SC} \times SD \times P}{A \times SA^2 + B \times SB^2 + C \times SC^2}$; and ſince the quantities of motion or moments generated in bodies in a given ‖ time, are as the quantities of matter and the forces which accelerate them jointly, the ſum of the moments generated in a given time will be expreſſed by $\dfrac{\overline{A \times SA + B \times SB + C \times SC} \times P \times SD}{A \times AS^2 + B \times SB^2 + C \times SC^2}$.

† Prop. III.

‖ Sect. I. Prop. IX.

It is manifeſt, that although the bodies A, B and C, and the quantities P and SD remain unaltered in quantity, yet the ſum of the moments generated in a given time will depend on the diſtances SA, SB, SC; but diminution and augmentation of theſe may be ſuch that the ſum of the moments generated in a given time may ſtill continue the ſame; there is therefore ſome intermediate diſtance at which if all the bodies be collected, the ſame moment will be produced by the force P acting for a given time, as when the bodies A, B, C are diſpoſed at their reſpective diſtances SA, SB, SC.

IV.

IV.

In a ſyſtem of bodies A, B, C moveable round an axis S, to which bodies motion is communicated by a force P acting at a given diſtance from S; it is required to aſſign ſome diſtance SO, at which if all the bodies are diſpoſed, the ſum of the moments generated in them in a given time ſhall be the ſame as before, every thing elſe remaining.

Let A, B and C be collected at the diſtance SO, then will the ſum of the moments generated in a given time be expreſſed by ‖ $\frac{\overline{A+B+C}\times SO\times P\times SD}{SO^2\times\overline{A+B+C}} = \frac{A\times AS+B\times BS+C\times CS}{A\times AS^2+B\times BS^2+C\times CS^2}\times P\times SD$ by the problem, and SO the diſtance required $= \frac{A\times AS^2+B\times BS^2+C\times CS^2}{A\times AS+B\times BS+C\times CS}$. Now it is manifeſt, that if the bodies A, B, &c. be transferred to the line SO, ‡ keeping their reſpective diſtances from S, the moments generated in a given time will be the the ſame as before. Let G be the common centre of gravity of A, B, C when collected in the line SO, and ſince in this caſe by the property of the centre of gravity $A\times SA+B\times SB+C\times SC = \overline{A+B+C}\times SG$, it follows, that $SO = \frac{A\times SA^2+B\times SB^2+C\times SC^2}{\overline{A+B+C}\times SG}$.

‖ Sect. VI. Prop. III. Cor. 1.

‡ Sect. VI. Prop. I. Cor. 6.

Fig. XXXVII.

Cor. 1. Thus let FGH repreſent a circle moveable in its own plane about an axis which paſſes through its centre S. Suppoſing the circle material, let the whole maſs be collected into any radius SC, the diſtance of each particle from S remaining the ſame; then will the denſity of the matter in the radius increaſe in the ratio of the diſtances from S. Let G be the centre of gravity of the weight thus diſpoſed SG is found by the uſual rule to be $\frac{2}{3}$ of SC. Moreover, if the particles thus condenſed in the line SC be denominated A, B, C, &c. the ſum of all the products $A \times AS^2 + B \times BS^2 + C \times CS^2$, &c. appears to be $\overline{A + B + C}$, &c. $\times \frac{SC^2}{2}$. We have therefore the diſtance SO as deſcribed in the problem $= \overline{A + B + C} \times \frac{SC^2}{2}$ divided by $\overline{A + B + C} \times \frac{2SC}{3}$, wherefore $SO = \frac{3SC}{4}$. Conſequently if the whole matter of the circle be collected into one or more points at the diſtance of $\frac{3}{4}$ of the radius from the axis, the ſame moment will be generated in a given time in the maſs thus diſpoſed, as if it were diffuſed uniformly over the whole area.

Cor. 2. In a given ſyſtem the ſum of the moments generated in the ſame time will be in a direct proportion of the diſtances at which the moving forces are applied from the axis; that is, if theſe forces be conſtant and every other circumſtance remain the ſame in both caſes.

In theſe propoſitions the effects of gravity on the bodies Fig. XXXVI. A, B and C have not been conſidered, motion being communicated to the ſyſtem by the action of a force applied at a given diſtance from the axis, and this force has been aſſumed of invariable quantity. When gravity acts upon the ſyſtem as a moving force, each particle contributes to the communication of motion, as well as to the inertia of the maſs moved; alſo the force of acceleration will be variable, unleſs the time of its action be taken evaneſcent: but in this caſe alſo the velocity generated in a ſyſtem in a given time, will be obtainable from the preceding rules, becauſe the force exerted by gravity on any body or ſyſtem of bodies will have preciſely the ſame effects to create motion (the maſs moved not being here conſidered) as if the whole weight were collected into the centre of gravity, and will be conſtant while the centre of gravity is deſcribing an evaneſcent arc.

Suppoſe therefore the bodies A, B and C to gravitate Fig. XXXVIII. towards the earth's centre, and let G be their common centre of gravity, and draw SGO; it is manifeſt that the point G will not be quieſcent, except when it is placed

|| Fig. XXXIX.
‡ Fig. XL.

coincident with its loweſt point, in which caſe the line SG is perpendicular || to the horizon: let it be moved out of that ſituation, ſuppoſe into the poſition ‡ SBG: with the centre S and diſtance SG deſcribe the arc GG and GD equal to it, and draw GB perpendicular to SBG; then the weight of the ſyſtem being denoted by the line SCG in quantity, and the direction in which gravity acts, let this be reſolved into two, i. e. SB which urges the ſyſtem directly from S, and BG perpendicular to SB which impels it round the axis of motion: in this ſituation therefore the point G will be acted on by a force in the direction BG, which is to the weight of the ſyſtem as BG to SG. This force will decreaſe continually as SBG approaches SCG, and when it coincides with SCG will be evaneſcent: the centre of gravity having then acquired its greateſt velocity will begin to aſcend on the contrary ſide, being retarded according to the ſame laws by which it was before accelerated, until it has loſt all motion at D; this motion of the ſyſtem round an axis through the angle GSD, when effected by gravity, is called a vibration or oſcillation, and the vibrating ſyſtem is called a pendulum.

It is next to be ſhewn, at what diſtance from the axis the whole weight of the ſyſtem muſt be collected, ſo that it ſhall vibrate through a given angle in the ſame time as before.

V.

Fig. XLI.

In a ſyſtem of bodies A, B, C, moveable round an horizontal axis which paſſes through S: let G be their common centre of gravity, when referred to a vertical plane paſſing through S, and in the line SG produced, let a point O be ſuppoſed ſuch, that if $A + B + C$ be collected into O, the ſyſtem will perform its vibrations through a given angle in the ſame time, as when the bodies A, B, C are diſpoſed at the diſtances S A, S B and S C reſpectively, and in their original poſitions in reſpect of each other; it is required to aſſign the diſtance S O.

Here two pendulums are to be conſidered, the one conſiſting of the bodies $A+B+C$, diſpoſed at their reſpective given poſitions and diſtances from the axis SA, SB, SC, the other being formed by collecting the whole maſs $A+B+C$ into ſome point O; and it is required by the problem to aſſign the diſtance SO, when theſe two pendulums perform the whole and every correſponding part of their vibration through a given angle in the ſame time. Let each pendulum begin its vibration from the line $SGNO$, and with the centre S and diſtances SG, SO deſcribe the arcs GgG, OoO, and through g which is contiguous to G draw the line So: moreover, draw the right ſine GM. Then during the time in which any point G at a given diſtance from the axis in both pendulums deſcribes the evaneſcent ‡ arc Gg, the force which accelerates that point will be conſtant in both pendulums, and if theſe forces are equal to each other at the firſt inſtant of motion, the point G in both pendulums ‖ will deſcribe the given arc Gg, and any ſubſequent § given increment of the arc GgG in the ſame elementary particle of time, and conſequently that whole arc in the ſame time. We have therefore only to aſcertain at what diſtance SO, the bodies $A+B+C$ muſt be collected, ſo that the point G may be accelerated by the ſame force as when the pendulum conſiſted of the bodies $A+B+C$ vibrating at their reſpective given poſitions and diſtances SA, SB, SC.

‡ Sect. III. Prop. IV. Cor. 4.

‖ Page 202. and Sect. III. Prop. IV.

§ Page 21. and 24.

The moving force which acts upon † the point G (being the centre of gravity of the firſt mentioned pendulum) is $\overline{A+B+C} \times \frac{MG}{SG}$, and this force will in a given time generate the ſame velocity in G, if A, B, C were removed, and the equivalent ‡ maſſes $\frac{A \times SA^2}{SG^2} + \frac{B \times SB^2}{SG^2} + \frac{C \times SC^2}{SG^2}$ were collected in G, the force therefore which accelerates the point *$G = \frac{\overline{A+B+C} \times MG \times SG}{A \times AS^2 + B \times BS^2 + C \times CS^2}$.

† Page 201.

‡ Sect. VI. Prop. I.

* Sect. VI. Prop. I. Cor. 3.

This is the caſe when the parts of the ſyſtem *A*, *B* and *C* are diſpoſed at their reſpective diſtances and poſitions SA, SB, SC: now let $A+B+C$ be collected in ſome point O, and through O draw ON perpendicular to SGO, the force which accelerates O will be $= \frac{\overline{A+B+C}\times ON}{\overline{A+B+C}\times SO} = \frac{GM}{SG}$, and the force which ‡ accelerates $G = \frac{MG}{SO}$.

‡ Sect. VI. Prop. II.

If the forces which accelerate the point *G* in both caſes be made equal, the abſolute velocity generated in equal particles of time in *G* will be the ſame in both pendulums, which will therefore deſcribe the given * arc of vibration GgG in the ſame time. Since therefore the forces which accelerate the point *G* in the two caſes deſcribed in the problem are to be equal, we have $\frac{MG}{SO} = \frac{\overline{A+B+C}\times MG\times SG}{A\times AS^2+B\times BS^2+C\times CS^2}$, and $SO = \frac{A\times AS^2+B\times BS^2+C\times CS^2}{\overline{A+B+C}\times SG}$.

* Sect. III. Prop. IV.

Cor. 1. The point *O* is called the centre of oſcillation, and the diſtance intercepted between the axis and this point determines the length of the pendulum.

Cor. 2. The whole maſs therefore which is contained in a pendulum may be ſuppoſed, as far as regards the time of its vibration through a given angle, to be collected into the centre of oſcillation.

Cor. 3. The ſum of all the products, which are formed by multiplying each particle of a ſyſtem into the ſquare of its diſtance from the axis of motion, is equal to the rectangle under the diſtances of the centres of gravity and oſcillation from the axis of motion, into the weight of the whole maſs, that is, if $A+B+C$, &c. $=W$, $A\times AS^2 + B\times BS^2 + C\times CS^2$, &c. $= SO\times SG\times W$.

Cor. 4. In theſe demonſtrations the bodies *A*, *B* and *C* which compoſe the revolving ſyſtem have been aſſumed of evaneſcent magnitude; and the rules demonſtrated are eaſily applied to the determination of the centres of oſcillation in natural bodies, which are made up of innumerable elementary particles, by finding the ſum of all the products which can be formed by multiplying each particle into the ſquare of its diſtance from the axis of motion, and dividing this ſum by the bodies whole weight into the diſtance of the common centre of gravity from the axis.

Cor. 5. When the revolving ſyſtem conſiſts of ſeveral bodies, the centres of oſcillation and gravity in each of which when revolving round the common axis is known, the centre of oſcillation of the whole ſyſtem is determined thus. Let $\overset{\prime}{O}, \overset{\prime\prime}{O}, \overset{\prime\prime\prime}{O}$ repreſent the diſtances of the centres of oſcillation of the ſeparate parts, from the axis, and $\overset{\prime}{G}, \overset{\prime\prime}{G}$, and $\overset{\prime\prime\prime}{G}$ be the diſtances of their reſpective centres of gravity; alſo let $\overset{\prime}{W}, \overset{\prime\prime}{W}, \overset{\prime\prime\prime}{W}$ be the weights of the ſeparate parts, and SG the diſtance of their common centre of gravity from the axis; then the diſtance of the centre of oſcillation of the whole ſyſtem from this axis will be

$$\frac{\overset{\prime}{O}\overset{\prime}{G}\overset{\prime}{W}+\overset{\prime\prime}{O}\overset{\prime\prime}{G}\overset{\prime\prime}{W}+\overset{\prime\prime\prime}{O}\overset{\prime\prime\prime}{G}\overset{\prime\prime\prime}{W}}{SG\times\overline{\overset{\prime}{W}+\overset{\prime\prime}{W}+\overset{\prime\prime\prime}{W}}}.$$

Cor. 6. Suppoſe the weight of a pendulum to be collected into the centre of oſcillation O, and let the pendulum perform its vibration in the arcs of a circle. The force which accelerates the pendulum when at P, will be to the force which accelerates it when at any other point Q as the ſines OM, ON. If theſe ſines were in the ſame proportion as the arcs OP, OQ, the * arcs would be deſcribed in the ſame time, provided the vibrations commenced from the points Q and P; but ſince the ſines increaſe not in ſo great a proportion as the arcs, it is manifeſt, that the time in which the pendulum deſcribes the larger arc of its vibration, will be greater than that in which the ſmaller vibrations are performed. But when the arcs are very ſmall, the ratio of the ſines becomes more nearly coincident with that of the arcs themſelves, and conſequently the forces which accelerate the pendulum are more nearly in that proportion, and accurately ſo when the arcs are evaneſcent. The times therefore wherein a pendulum performs its leaſt vibrations in circular arcs will be equal, whatever be the proportion of thoſe arcs, and may be determined by referring to prop. 1. ſect. 111. Suppoſe SO to be perpendicular to the horizon, being the quieſcent poſition of the pendulum, and let it be deflected from its vertical ſituation through an arc PO, then will the force accelerating the point P, be that part of gravity which is expreſſed by the fraction $\frac{OM}{SO}$, and when the arc PO is

Fig. XLII.

* Sect. III. Prop. I.

evaneſcent, the force $= \frac{PO}{SO}$; let $p = 3.14159$, $l = 193$, then will the time wherein the pendulum deſcribes OP

‡ Sect. III. Prop. I. Cor. 3.

$= \frac{p \times \sqrt{PO}}{\sqrt{8 \frac{PO}{SO} \times l}} = \frac{p \times \sqrt{SO}}{\sqrt{8l}}$ ‡, and the time of one entire vibration $= \frac{p \times \sqrt{SO}}{\sqrt{2l}}$ parts of a ſecond.

Fig. XLIII. Let ABC be ſyſtem of bodies without weight, moveable about an axis which paſſes through S. Let $So = \frac{A \times AS^2 + B \times BS^2 + C \times CS^2}{A \times AS + B \times BS + C \times CS}$, then if the whole matter were collected in o, the ſame ‖ quantity of motion would be generated in the ſyſtem in a given time, as when $A + B + C$ revolved at their reſpective diſtances SA, SB, SC: if any given moving force W, by which the angular motion is generated, be applied at the ſame diſtance from the axis SG in both caſes, and in a direction perpendicular to SG; the ſum of the moments generated in a given time will in this caſe be expreſſed by the quantity $\frac{SG \times W}{So}$.

‖ Sect. VI. Prop. IV.

Now let the whole maſs $A + B + C$ be collected into any other point O, and the force W be applied in a direction perpendicular to SO to communicate motion to the point O, then will the quantity of motion generated in a given time be denoted by W, and the proportion of theſe quantities of motion generated in a given time will be that of SG to So. This is applicable to the vibratory motion of a pendulum which is effected by gravity. The force to create motion in pendulums is the gravity of each particle, and this will be equivalent in its effect to the entire weight applied at the centre of gravity: therefore, let G be the centre of gravity, and $So = \frac{A \times AS^2 + B \times BS^2 + C \times CS^2}{A \times AS + B \times BS + C \times CS}$, and let O be the centre of oſcillation: the quantity of motion generated in the pendulum by gravity in a given time will be to the quantity of motion, which would be generated in it when the whole maſs is collected into the centre of oſcillation, in the proportion of $SG : So$.

VI.

In a given ſyſtem of bodies A+B+C if Fig. XLIV. the diſtance of the centre of gravity from the axis of motion be increaſed, the diſtance between the centres of gravity and oſcillation will be decreaſed in the ſame proportion, provided the plane wherein the pendulum vibrates be the ſame in reſpect of the bodies A, B and C.

Let S be the point through which the axis of motion paſſes; alſo let G be the centre of gravity, and O the centre of oſcillation, and draw SGO; join GA, GB and GC, and through A, B and C draw the lines Aa, Bb, Cc perpendicular to SO: then we have $AS^2 = GS^2 + AG^2 + 2SG \times Ga$, $BS^2 = GS^2 + BG^2 + 2SG \times Gb$, $CS^2 = GS^2 + CG^2 - 2SG \times Gc$, (becauſe $CS^2 = CG^2 - GS^2 - 2GS \times Sc = CG^2 + GS^2 - 2GS^2 - 2GS \times Sc = CG^2 + GS^2 - 2SG \times \overline{GS + Sc}$) and $A \times AS^2 + B \times BS^2 + C \times CS^2 = A \times \overline{GS^2 + GA^2 + 2SG \times Ga} + B \times \overline{GS^2 + GB^2 + 2SG \times Gb} + C \times \overline{GS^2 + GC^2 - 2SG \times Gc}$: but by the property of the centre of gravity, $A \times Ga + B \times Gb - C \times Gc = 0$, wherefore $A \times AS^2 + B \times BS^2 + C \times CS^2 = \overline{A + B + C} \times GS^2 + A \times AG^2 + B \times BG^2 + C \times CG^2$, and Euclid. II. Lib. Prop. 12.

$$SO = \frac{\overline{A+B+C} \times GS^2}{\overline{A+B+C} \times GS} + \frac{A \times AG^2 + B \times BG^2 + C \times CG^2}{\overline{A+B+C} \times GS}$$

$= GS + GO$, and taking GS from both ſides, we have $GO = \dfrac{A \times AG^2 + B \times BG^2 + C \times CG^2}{\overline{A+B+C} \times SG}$, and $\dfrac{A \times AG^2 + B \times BG^2 + C \times CG^2}{\overline{A+B+C}\ \text{[illegible]}}$ being conſtant in the ſame ſyſtem, when the plane of vibration alters not, it follows, that GO will be inverſely proportional to SG.

Cor. 1. With the centre G and diſtance SG, deſcribe a circle SRF. Let RQ be the diſtance of the centre of oſcillation from the axis of motion when the pendulum vibrates in the ſame plane as before, round any point in the circumference SRF, for example R, then will $RQ = SO$, for $GQ = \frac{A\times AG^2 + B\times BG^2 + C\times CG^2}{GR\times A+B+C} =$ GO by the propoſition, and conſequently $SO = RQ$: the ſame ſyſtem will therefore perform its vibrations in a given arc of a circle in the ſame time, wherever its centre of ſuſpenſion be ſituated, provided it be always at the ſame diſtance from the centre of gravity, and the plane of its vibration in reſpect of the ſyſtem be not altered.

Fig. XLV. Cor. 2. If G be the common centre of gravity of the whole ſyſtem, and of any two or more parts, for example A and B, then the diſtance BA remaining, let theſe bodies revolve round G, in the plane which is perpendicular to the axis of motion; wherever they are fixed during this revolution, the ſyſtem will vibrate through a given angle round the axis of motion S in the ſame time as before; for by the revolution of AB, the quantity $A\times AG^2 + B\times BG^2 + C\times CG^2$ is not altered, and SG being likewiſe the ſame, $GO = \frac{A\times AG^2 + B\times BG^2 + C\times CG^2}{\overline{A+B+C}\times SG}$ will be unvaried, and conſequently SO the ſame as before.

Fig, XLIV. Cor. 3. Let A, B, C repreſent any ſyſtem revolving round an axis paſſing through S; let G be the centre of gravity, O the centre of oſcillation, and W the weight of the body; then if the ſyſtem when referred to any one plane, which is perpendicular to the axis, conſiſts of the particles A, B, C, &c. the ſum of the products which are formed by multiplying each particle into the ſquare of its diſtance from the centre of gravity $= SG \times GO \times W$: that is, when theſe diſtances are eſtimated by lines drawn through each particle perpendicular to a line which paſſes through the centre of gravity; or if the whole figure be projected into a plane perpendicular to the axis and paſſing through the centre of gravity, theſe diſtances may be eſtimated by lines intercepted between the particles and the centre of gravity itſelf.

Fig. XLVI. Cor. 4. Let ABC repreſent the plane which paſſes through the centre of gravity of any irregular ſolid, and let a line be drawn perpendicular to this plane paſſing through the centre of gravity G. Any where in the plane $SABC$, let two points SB found by trial be ſuch, that when they are ſupported, the ſolid will remain balanced in any poſition, the centre of gravity muſt be in a line which joins theſe points. Let the body vibrate in the plane $SABC$ round an horizontal axis which paſſes through S. Elevate the point B ſo that SB may be horizontal, and let the weight q acting vertically by means of a fixed pully C be ſuch as is juſt ſufficient to ſupport SB horizontal: alſo let $SB = a$, the weight of the body $= w$, then will $SG = \frac{aq}{w}$. Suppoſe that the ſyſtem performs n very ſmall vibrations in a circular arc round the given axis which paſſes through S in t ſeconds; then the diſtance of the centre of oſcillation from the axis $SO = \frac{2t^2l}{n^2p^2}$ inches*, l being $=$ 193 inches, and $p = 3{\cdot}14159$, &c. wherefore $GO = \frac{2t^2l}{n^2p^2} - \frac{aq}{w} = \frac{2wt^2l - n^2p^2aq}{n^2p^2w}$, and † $w \times GO \times SG = A \times AG^2 + B \times BG^2 + C \times CG^2$, &c. $= \frac{2waqt^2l - a^2q^2n^2p^2}{n^2p^2w}$. This propoſition is of conſiderable uſe in the practical conſtruction of the balance, when the motion of that inſtrument as well as its equilibrium is conſidered, which will be ſhewn on a future occaſion.

Fig. XLVII. Suppoſe, that during the vibration of a ſyſtem of bodies round a fixed axis, ſuch an obſtacle were oppoſed to any point O, as to entirely deſtroy the motion of that point. The point O being quieſcent, each particle of the ſyſtem will endeavour by its inertia to proceed in the direction of its ‡ motion, that is, of the tangent to the circular arc which it was deſcribing the inſtant O was ſtopped. Theſe forces will therefore act on the ſyſtem to turn it round O, and if the ſum of the forces on each ſide of O ſhould be unequal, the motion of the ſyſtem will not be deſtroyed when O is ſtopped: but ſince the forces which act on the pendulum between O and S will have an effect to continue the motion of the ſyſtem contrary to thoſe which are impreſſed on the other ſide of O, if the point O be ſo ſituated that the ſum of the forces to turn the ſyſtem round O on each ſide of that point may be exactly equal, the whole motion of the ſyſtem will be deſtroyed the inſtant O is ſtopped. This point, determined according to the conditions juſt deſcribed, is called the centre of percuſſion, and if a pendulum vibrating with a given angular motion ſtrikes an obſtacle, the effect of the blow will be the greateſt when the impact is made in that point: for in

* Sect. III. Prop. I.

† Sect. V. Prop. I. Cor. 3.

‡ I. Law of Motion.

this caſe, the obſtacle receives the whole revolving motion of the pendulum; whereas if the blow be ſtruck in any other point, a part of the pendulum's motion will be employed in endeavouring to continue its rotation.

VII.

Fig. XLVII. In a given ſyſtem of bodies A + B + C moveable about an axis of motion paſſing through S; it is required to determine the diſtance of the centre of percuſſion from the axis of motion.

Let G be the common centre of gravity, draw $SGOL$, and let O be the centre of percuſſion. Through A, B and C draw Aa, Bb and Cc perpendicular to SA, SB and SC reſpectively, and the lines AD, BE, CF perpendicular to SO: then the inſtant O is ſtopped, the body A will endeavour to proceed in the direction Aa, being perpendicular to SA, and thereby urge forward the point a with a force proportional to $A \times SA$; this may be expounded in quantity and direction by the line Aa, but the direction in which it acts being oblique to SO, in order to find what part of it is employed to turn the ſyſtem round O, it muſt be reſolved into two Da and DA, whereof Da tends to urge the ſyſtem from S, and AD to impel it round O: conſequently this latter force is that part of the whole force $A \times SA$, which is expreſſed by the fraction $\frac{DA}{Aa}$, or becauſe of the ſimilar triangles DAa, SDA, by the fraction $\frac{SD}{SA}$: the force of A therefore applied to turn the ſyſtem when O is ſtopped $= \frac{A \times AS \times SD}{SA} = A \times SD$, and the effect of this force to generate angular motion round O is $A \times SD \times Oa = A \times SD \times \overline{SO - Sa}$; and ſince by ſimilar triangles * $Sa = \frac{SA^2}{SD}$, it follows, that the force of A to turn the ſyſtem round $O = A \times SD \times SO - A \times SA^2$; and

* Euclid. Elem. Lib. 6. Prop. 13.

and by the ſame reaſoning the force of B to turn the ſyſtem round O in a contrary direction $= B \times SB^2 - B \times SE \times SO$, and that of $C = C \times CS^2 - C \times CF \times SO$: and ſince theſe forces are to balance each other round O by the problem, we have $A \times SD \times SO - A \times SA^2 = B \times SB^2 - B \times SE \times SO + C \times SC^2 - C \times SF \times SO$, and $SO = \frac{A \times SA^2 + B \times SB^2 + C \times SC^2}{A \times SD + B \times SE + D \times SF}$; and ſince by property of the centre of gravity $A \times SD + B \times SE + C \times SF = SG \times \overline{A + B + C}$, it follows, that $SO =$

$$\frac{A \times AS^2 + B \times BS^2 + C \times CS^2}{\overline{A + B + C} \times SG}.$$

Cor. 1. The diſtance* of the centre of percuſſion is equal to the diſtance of the centre of oſcillation from the axis of motion. * Sect. VI. Prop. V.

The centre of oſcillation has been defined that point wherein if all the matter of a ſyſtem were collected, the time of vibration through a given angle would be the ſame as before, and conſequently the angular velocity generated in a given time muſt be the ſame in both caſes: but here a circumſtance is to be taken notice of. In determining the centre of oſcillation, when the parts of a Fig. XLI. ſyſtem A, B, C are diſpoſed at the reſpective given poſition and diſtances SA, SB, SC, the moving force which generates the angular velocity, that is, the natural weight of the body is applied at the common centre of gravity; whereas, when the matter of the ſyſtem is collected into O, ſo that the time of vibration in a given angle may be the ſame as before, the moving force is applied at O, having changed its place from G to O.

This is the caſe when a pendulum is put in motion by gravity, but when any ſyſtem of bodies A, B, C is cauſed to revolve by the action of a force applied at any given diſtance which changes not, if the whole maſs $A + B + C$ be collected into the centre of oſcillation, the angular velocity generated in a given time will not be the ſame as before: there exiſts therefore ſome other point R, in which Fig. XLIII. if all the matter $A + B + C$ be collected, a given force P applied at an invariable diſtance from the axis, will generate the ſame angular velocity in the ſame time as if the bodies were diſpoſed at their reſpective diſtances SA, SB, SC. This point is called the centre of gyration.

N. B. To avoid repetitions, whenever any moving force is applied to turn a ſyſtem round its axis, it is always meant to act in the direction of the plane wherein the ſyſtem revolves, and at right angles to a line, drawn from the point at which it is applied, perpendicular to the axis.

VIII.

Fig. XLIII.

In a ſyſtem of bodies A, B, C revolving round an axis which paſſes through S: having given the quantities of matter contained in A, B, C, and the diſtances SA, SB, SC, it is required to determine SR the diſtance of the centre of gyration from the axis of motion.

Let any force P be applied to turn the ſyſtem at any given diſtance from the axis SD. If the force which accelerates a given point D be the ſame in any two caſes, the *abſolute velocity of that point generated in a given time muſt be the ſame, and conſequently the angular velocity ||generated in the ſyſtem equal in both caſes: the force which accelerates the point D = $\frac{P\times SD^2}{A\times AS^2+B\times BS^2+C\times CS^2}$:† now, let $A+B+C$ coincide with R, and the force which accelerates D will be $\frac{P\times SD^2}{\overline{A+B+C}\times SR^2}$, theſe forces muſt be equal by the hypotheſis: making therefore $\frac{P\times SD^2}{A\times AS^2+B\times BS^2+C\times CS^2}$ $= \frac{P\times SD^2}{\overline{A+B+C}\times SR^2}$, we ſhall have $SR^2 =$ $\frac{A\times AS^2+B\times BS^2+C\times CS^2}{A+B+C}$, and the diſtance of the centre of gyration required $= \sqrt{\frac{A\times AS^2+B\times BS^2+C\times CS^2}{A+B+C}}$.

* Sect. III. Prop. I.

|| Sect. VI. Prop. I. Cor. 4.

† Sect. VI. Prop. I. Cor. 3.

Cor. 1. Let $A+B+C=W$, and ſince $A\times AS^2+B\times BS^2+C\times CS^2=SR^2\times W$, it follows, that SR^2 =

$= SO \times SG$: † wherefore the diſtance of the centre of gyration is a mean proportional between the diſtances of the centres gravity and oſcillation from the axis of motion.

† Sect. VI. Prop. VI. Cor. 3.

Cor. 2. Every thing remaining, the angular velocity generated in a ſyſtem will be the ſame, whether the maſs W is placed at the diſtance ‖ $SR = \sqrt{SO \times SG}$, or whether the maſs $\frac{W \times SR^2}{SD^2}$ is placed * at the diſtance SD.

‖ Supra.

* Sect. VI. Prop. I.

Cor. 3. Let the ſyſtem revolve in the ſame plane as before, and round an axis which paſſes through the centre of gravity; and let $GQ = \sqrt{\frac{A \times GA^2 + B \times GB^2 + C \times GC^2, \&c.}{A+B+C}}$ being the diſtance of the centre of gyration ‡ from the axis of motion: from the propoſition it appears, that $GQ^2 = SG \times GO$.

‡ Sect. VI. Prop. I.

Cor. 4. Let $ABCD$ repreſent any pendulous body moveable round an axis of motion HI which paſſes through S, and let any impact be impreſſed on the point D, (while the pendulum is at reſt,) in an horizontal direction and perpendicular to the vertical plane $ABCD$. To find the angular velocity, which is communicated to the pendulum by the impact, let the weight of the pendulum be W, let G be the centre of gravity, O the centre of oſcillation, and R the centre of gyration; then let a weight be aſſumed which is to W as SR^2, or $SO \times SG$ to SD^2, this weight $= \frac{W \times SO \times SG}{SD^2}$ being collected into the point D, the ſame angular velocity will be generated in the ſyſtem, as when the pendulum is of the prior form, every thing elſe being the ſame.

Fig. XLVIII.

This is the rule delivered in Mr. Robins' Gunnery, without demonſtration, page 86. vol. 1. and by the application of it to his experiments, Mr. Robins was the firſt who aſcertained with any tolerable degree of certainty the initial velocity of military projectiles.

Gallilæo had demonſtrated that the track of a body projected in any direction not perpendicular to the horizon, was a parabola, the air's reſiſtance being not conſidered, and this reſiſtance was imagined to be ſo ſmall, that bodies moving through it would not ſenſibly deviate from the path which they would deſcribe in vacuo, the velocity and direction of projection being the ſame. § This opinion however erroneous was adopted by Dr. Hally, as well as ſeveral

§ Philoſ. Tranſact. No. 129.

veral other eminent philoſophers. That either the theory of of projectiles was defective, or the reſiſtance of the air ſuch as rendered it inapplicable to practice, was however afterwards diſcovered. When equal bullets are ſhot off from the ſame piece of ordnance, and with equal charges and at the ſame elevation above the horizon, the horizontal ranges and times of flight muſt be the ſame or very nearly ſo, (provided the figure of the balls be perfectly ſpherical, and their ſubſtance altogether homogeneal,) however great the air's reſiſtance may be. This being the caſe, if the horizontal range were obſerved on any one diſcharge at a given angle of elevation, the initial velocity wherewith the ſhot quitted the cannon would be deduced from Galilæo's rules, provided the air's reſiſtance were inconſiderable in its effects. But it appeared from experiments made on the ſame piece of ordnance from which bullets exactly equal were diſcharged with equal quantities of powder and at different elevations, that the initial velocities deduced by Galilæo's theory from the different horizontal ranges obſerved were entirely different. Neither did the horizontal ranges follow that law in reſpect of the elevations, which is demonſtrated by Galilæo.

Various hypotheſes were formed in order to account for theſe irregularities. Mr. Robins perceived that their true cauſe could not be determined, except the initial velocity of the ſhot in given circumſtances were firſt with certainty known. The great velocities communicated to bullets by gunpowder, rendered it impoſſible to make any direct obſervation on them : Mr. Robins therefore uſed the following indirect method, which ſeemed fully to anſwer his intentions.

Fig. XLVIII. *SABCD* repreſents a pendulum moveable about an horizontal axis *IH*: a heavy block of wood *ABCD* was ſo affixed to the rod *ES*, that its anterior ſurface might be vertical when quieſcent and parallel to a plane which paſſed through the axis of motion. When a bullet impinged perpendicularly on the ſurface *AD*, the pendulum was cauſed to vibrate on its axis of motion, and the weight of it was ſo adjuſted in reſpect of the bullet's weight and velocity, that the angle of vibration might be ſuch as admitted of the moſt exact determination by experiment.

Suppoſe then that a muſket or other bullet were to ſtrike perpendicularly againſt the point *D*, and the angle through which the pendulum is impelled by the ſhot obſerved. Mr. Robins' problem is to infer the velocity with which the bullet impinged againſt the pendulum from having given this angle, together with the dimenſions and weight of the pendulum: the ſolution of this may ſerve as an illuſtration of the principles of rotation juſt demonſtrated.

IX.

Fig. XLVIII.

Let a muſket or other ball perpendicularly ſtrike againſt and penetrate into the ſubſtance of the pendulum ABCD in the point D; let the angle through which the pendulum is impelled by the ſhot be A°: having given the weight of the pendulum and of the ball, and the diſtance from the axis of the centres of oſcillation and gravity of the pendulum together with the diſtance SD; it is required to aſſign the velocity with which the ball ſtruck the pendulum.

Let the weight of the pendulum $= W$, and that of the ball $= p$; alſo, let O be the centre of oſcillation, G the centre of gravity, R the centre of gyration, and let Q be a weight which being collected into the point D, a given force applied to D will generate the ſame angular velocity as if it were applied againſt the pendulum itſelf in the point D. This ||equivalent weight $Q = \frac{W \times SG \times SO}{SD^2} = \frac{W \times SR^2}{SD^2}$ by the laſt propoſition: alſo, let V be the velocity communicated to the point D, x the velocity of the bullet required, then the block of wood being nonelaſtic the laws of colliſion give the following proportion $Q + P : P :: x : V$; wherefore the velocity of the ball $x = V \times \frac{Q + P}{P}$. Conſequently ſince Q and P are known, it only

|| Sect. VI. Prop. VIII. Cor. 4. and Prop. I.

remains to aſſign the velocity V which has been communicated to the point D, from having given the angle $A°$, through which the pendulum is impelled by the ſtroke. Let $SO = s$, $SD = d$, then the velocity acquired by the centre of oſcillation in a pendulum, which deſcribing from reſt any arc of a circle has arrived at its loweſt point, is equal to that acquired by a heavy body which has deſcended freely from reſt by the acceleration of gravity, through a ſpace equal to the verſed ſine of the arc deſcribed by the pendulum: In like manner, if any given velocity be communicated to the centre of oſcillation of the pendulum when quieſcent, it will riſe through an arc whoſe verſed ſine is equal to the ſpace through which a body falls freely from reſt in order to acquire that velocity. This being the caſe, ſince the pendulum in the problem deſcribes an arc $A°$, let b be the verſed ſine of the arc $A°$ to radius $= 1$, then will the centre of oſcillation or the point O deſcribe an arc during its motion, the verſed ſine of which $= sb$, and conſequently if $l = 16\frac{1}{12}$ feet, the initial velocity* of O will be $= \sqrt{4lbs}$ feet in a ſecond, and the velocity of the point $D = \sqrt{\frac{4lbd^2}{s}} = V$. This therefore will give us the velocity of the bullet the inſtant it ſtruck the pendulum or

$$x = V \times \frac{Q+P}{P} = \sqrt{\frac{4lbd^2}{s}} \times \frac{Q+P}{P} \text{ feet in a ſecond.}$$

To illuſtrate this, let an example be aſſumed from Mr. Robins. The pendulum which he made uſe of weighed 56 pound 3 oz. or 675 oz. The diſtance of the impact from the axis $= 66$ inches $= SD$, $SO = 62.66 = s$, $SG = 52$. Mr. Robins † after having deſcribed the conſtruction of his inſtrument and the conditions of this problem, ſays, " In the compound ratio of 66 to $62\frac{2}{3}$, and 66 to " 52, take the quantity of matter in the pendulum to a " fourth quantity, which is 42 pounds $\frac{1}{2}$ oz. Now geo" meters will know, that if the blow be ſtruck in the " centre of the piece of wood *ABCD*, the pendulum will " reſiſt the ſtroke in the ſame manner, as if this quantity " of matter only (42 pounds $\frac{1}{2}$ oz.) was concentrated in " that point, and the reſt of the pendulum was taken " away."

This is manifeſtly the rule juſt ‡demonſtrated for a compound ratio of 66 to $62.\frac{2}{3}$, and 66 to 52 is the ſame as the

* Sect. VI. Prop. V. Cor. 3.

† Robins' Gunnery, Vol. I.

‡ Prop. VIII. Cor. 3.

ſquare of the diſtance of the impact to the rectangle under the diſtances of the centres of gravity and oſcillation from the axis of motion; that is, in a duplicate ratio of the diſtance of impact to that of the centre of gyration from the axis. The equivalent weight therefore which in the preceding ſolution was denoted by $\mathcal{Q}$ will $= W \times$

$$\frac{SG \times SO}{SD^2} = \frac{675 \times 62.66\frac{2}{3} \times 52}{66 \times 66} = 504.95 \text{ oz.}$$

In the experiment of Mr. Robins, the chord of the angle through which the pendulum was impelled‡ by the ſtroke, was obſerved to be 17.25 inches to a radius of 71.125 inches. This gives the angle itſelf $= 13^\circ\, 55'\, 48''$; we have moreover for the determination of the bullet's velocity of impact from the propoſition, $d = 66$, $s = 62\frac{2}{3}$ and $\mathcal{Q} = 42$ pounds $\frac{95}{100}$ of an oz. or 504.95 oz. $b =$ the verſed ſine of $13^\circ\, 55'\, 48''$ to radius $1 = .02941057$, $P = 1$ oz. and the bullet's velocity required $= \sqrt{\frac{4lbd^2}{s}} \times \frac{\mathcal{Q}+P}{P} = 39.727 \times 505.95 = 20100$ inches, or 1675 feet in a ſecond, l being $= 193$ inches: if according to Mr. Robins, l be aſſumed $= 196.06$, the velocity of the bullet will be $= 1688$. Mr. Robins† makes the velocity from the ſame data to be 1641 feet in a ſecond: or if the computation had been exact according to his principles, the velocity would have been 1645. This difference is remarked in the excellent comment* on Mr. Robins' Principles of Gunnery: it was occaſioned by his having aſſumed as true, that a given velocity being communicated to any point of the pendulum D when quieſcent, will cauſe that point to deſcribe an arc of which the verſed ſine is equal to the ſpace through which a body muſt fall from reſt by its gravity, to acquire that velocity: but this is true only when the given velocity is communicated to the centre of oſcillation: if therefore a given velocity be communicated to any other point D, in order to find to what perpendicular altitude that point will riſe, the velocity at the ſame time impreſſed on the centre of oſcillation muſt be firſt found; then the perpendicular altitude to which the centre of oſcillation riſes will be equal to the ſpace through which a body muſt fall freely by gravity to acquire that velocity, and from this the altitude to which the point ſtruck will riſe, may be inferred by the rule of pro-

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‡ Robins' Gunnery, page 87.

† Page 88.

* Euler's Comm. on Robins' Gunnery, p. 55. Engliſh Edit.

proportion. Thus let V be the velocity communicated to the point D‡, then will $V \times \frac{SO}{SD}$ be the velocity communicated to the centre of oſcillation, and if $l = 193$ inches, or $16\frac{1}{12}$ feet, the verſed ſine of the arc through which the centre of oſcillation is impelled* $= \frac{V^2 \times SO^2}{4lSD^2}$, wherefore the perpendicular altitude through which the point ſtruck D will aſcend $= \frac{V^2 \times SO}{4l \times SD}$; and conſequently if this altitude $= a$ by obſervation, V the velocity communicated will be inferred; for ſince $\frac{V^2 \times SO}{4l \times SD} = a$, we have $V = \sqrt{\frac{4l \times SD \times a}{SO}}$. Mr. Robins' rule makes it $\sqrt{4la}$, which will give the reſult too ſmall in the proportion of $\sqrt{SO} : \sqrt{SD}$, when SO is leſs than SD, and greater in the ſame proportion when SO is greater than SD: thus in the example juſt referred to $1645 : 1688 : \sqrt{62.\frac{2}{3}} : \sqrt{66}$.

‡ Fig. XLVIII.

* Sect. III. Prop. V. Cor. 3.

In this ſolution ſeveral minute circumſtances which may in ſome degree affect the velocity deduced from the rule are not taken into account, being in general inſenſible in experiment: ſuch as the reſiſtance of the air, the figure of the ball which has in the ſolution been ſuppoſed a point: alſo the ball's weight will alter the centres of oſcillation and gravity, and conſequently the equivalent weight Q. Moreover, the time during which the ball penetrates the block is aſſumed as an inſtant, whereas it is of finite quantity, however ſmall: but theſe circumſtances are in general too minute to have any ſenſible effect: in particular caſes corrections may be applied, if they ſhould be neceſſary.

X.

Fig. XLIX. In a ſyſtem of bodies which may vibrate as a pendulum round any axis of motion perpendicular to the plane ABC: having given the centre of gravity G, draw any line SGOL through G in the plane of motion

ABC; let it be required to aſſign in this line a point Q, through which if the axis of motion paſſes, the pendulum ſhall vibrate in the leaſt time poſſible.

||Let O be the centre of oſcillation, Q the centre of ſuſpenſion, and W the weight of the pendulum: moreover, let the ſum of $A \times AG^2 + B \times BG^2 + C \times CG^2$, &c. $= S$: alſo, let $GQ = x$, $GO = y$; then $x + y$ will be the length of the pendulum which is to be a minimum by the problem, wherefore $\dot{x} + \dot{y} = 0$, and $\dot{x} = -\dot{y}$; but *ſince $xy = \frac{S}{W}$, taking the fluxions $x\dot{y} = -y\dot{x}$, in which equation ſubſtituting $-\dot{y}$ for $\dot{x}$, we have $x = y$, that is, $GQ = GO$, and $xy = \frac{S}{W} = \frac{A \times AG^2 + B \times BG^2 + C \times CG^2}{A + B + C} = GQ^2$, wherefore $GQ = \sqrt{\frac{A \times AG^2 + B \times BG^2 + C \times CG^2}{A + B + C}} =$ to the ‡ diſtance of the centre of gyration from the axis of motion, when that axis paſſes through the centre of gravity.

|| Bernoulli.

* Sect. VI. Prop. VI. Cor. 3.

‡ Sect. VI. Prop. VIII.

Let therefore the ſyſtem revolve round the axis which paſſes through the centre of gravity; and ſuppoſe R to be the centre of gyration when all the matter is collected into any line SGL, each particle keeping its reſpective diſtance from G, then with the centre G and diſtance GR deſcribe a circle, and if the ſyſtem vibrates in the ſame plane as before, the time of vibration will be leſs when the axis of ſuſpenſion paſſes through any point in the circumference of the circle juſt deſcribed, than when it is further from the centre of gravity or nearer to it.

Thus, let ABC repreſent a cylinder, if the maſs contained in that body were collected into the line GC, each particle preſerving the ſame diſtance from the axis as before, the centre of gyration R would be diſtant from the centre of the circle G by $\frac{1}{\sqrt{2}}$ the radius: if therefore the radius GC be divided into 100 parts, and GR be taken equal to about 71 of them, and the cylinder vibrates in its own plane round an horizontal axis paſſing through R, the time of vibration will be leſs than if R were placed at

Fig. L.

a greater or leſs diſtance from the centre of gravity. Thus alſo, if a right line AB or ſlender cylinder vibrates about a point R, and if $AB = 100$ parts, and AR about 21 of them, the line will vibrate about R in leſs time than if AR were greater or leſs, the angle through which the pendulum vibrates being the ſame in all theſe caſes.

Whenever the centre of gyration of any figure which revolves round an axis paſſing through the centre of gravity G, is mentioned, it is meant to be the centre of gyration of the body when collected into any given line in the plane of motion paſſing through the axis, and each particle keeping its reſpective diſtance from the axis.

This is here mentioned, becauſe when the axis of motion paſſes through the centre of gravity, $SG = 0$, and SO is infinite; but $SR^2 = SG \times SO = \dfrac{A \times GA^2 + B \times BG^2 + C \times GC^2}{A + B + C}$ alſo in this caſe, and may be determined either by finding the ſum of $A \times GA^2 + B \times BG^2$, &c. and dividing it by $A + B + C$, or by ſuppoſing the maſs to be collected into any line GD, which paſſes through the centre of gravity, each particle preſerving its reſpective diſtance from the axis. Let o be the centre of oſcillation of the line GD, when the maſs is ſo collected into it, and g its centre of gravity, then will $SR^2 = Sg \times So$ as before. The quantity SR being aſcertained, with the centre G and diſtance GR, let a circle be deſcribed: then the centre of gyration may be any where in the circumference of the circle, the angular velocity * generated in the ſyſtem in a given time depending on its diſtance from the axis only.

XI.

Fig. LI.

Let ABC repreſent a ſection or plane which paſſes through the centre of gravity of any irregular body. Suppoſe this body to vibrate as a pendulum in the plane ABC round an axis which paſſes through S; it is required to determine the diſtance from the centre of gravity of any other

* Sect. VI. Prop. I. Cor. 6.

other point Q, through which if the axis paſſes, the time in which the pendulum vibrates through a given angle ſhall be the ſame as before, the plane of vibration not being altered.

Let R be the centre of gyration of the ſyſtem, when it revolves round an axis which paſſes through the centre of gravity, and take a third proportional GQ to the lines GS, GR. With the centre G, and diſtances GS, GQ deſcribe the circles LSV, LQV: if the axis of the pendulum paſſes through any point in the circumferences of theſe two circles, the diſtance of the centre of oſcillation from the axis of motion will be always equal to the ſum of the radii of the circles, the time of vibration therefore through a given angle will be the ſame in both caſes.

Let the pendulum vibrate round S, and let O be its centre of oſcillation: then ſuppoſing the particles which compoſe the ſyſtem when referred to the plane of vibration to be A, B, C, &c. the ſum of all the products $A \times AG^2 + B \times BG^2 +$ ‖ $C \times CG^2$, &c. $= GR^2 \times \overline{A+B+C} = SG \times GO$ ‡ $\times \overline{A+B+C}$, &c. wherefore $GO = \frac{GR^2}{SG}$, and $SO = GO + SG = \frac{GR^2}{SG} + SG = \frac{GR^2 + SG^2}{SG}$; but by conſtruction $GR^2 = SG \times QG$, wherefore $SO = \frac{SG \times QG + SG^2}{SG} = QG + SG =$ to the ſum of the radii. Now let the pendulum vibrate in the ſame plane round Q, and let H be its centre of oſcillation, then will † $GH = \frac{GR^2}{GQ}$, and $QH = GQ + \frac{GR^2}{GQ}$: but $GR^2 = GS \times GQ$ by conſtruction, wherefore by ſubſtitution $QH = GQ + \frac{GS \times GQ}{GQ} = GQ + GS$, or the ſum of the circle's radii as before.

Cor. 1. It appears that there are two diſtances of the axis of motion from the centre of gravity of any pendulum, either of which being aſſumed, the time wherein the

‖ Sect. VII Prop. VI. Cor. 3.

‡ Sect. VI. Prop. VIII.

† Sect. VI. Prop. VIII. Cor. 3.

pendulum vibrates through a given arc will be the ſame, the plane of vibration not being altered.

Cor. 2. When the axis of motion paſſes through the centre of gyration R, there is no other diſtance of the axis of motion from the centre of gravity, which will cauſe the pendulum to vibrate in the ſame time, the two diſtances before determined now coinciding with GR; and in this caſe the * pendulum vibrates in a given angle in the leaſt time poſſible. * Sect. VI. Prop. X.

Cor. 3. In any pendulum which vibrates in a given plane if an axis of motion be made to paſs through the point which was before the centre of oſcillation, the former point through which the axis paſſed will become the centre of oſcillation.

XII.

Fig. LII. In a ſyſtem moveable round a fixed axis, and conſiſting of bodies the weights and figures of which are known: having given the moving force P, and the diſtance from the axis SD, at which it is conſtantly applied, let it be required to determine the force which accelerates D.

Let the moving force be a weight P applied ſo as to act on D by means of a line wound round the circumference DFE: whatever be the figures of the bodies which compoſe the ſyſtem, if the component particles be $A+B+C$, &c. $+E+F+G$, &c. the force which accelerates D will be $=$

$$\frac{P \times SD^2}{P \times SD^2 + A \times AS^2 + B \times BS^2, \&c. + E \times ES^2 + F \times FS^2, \&c.}$$

but in any body which revolves round a fixed axis S, if SO be the diſtance of the centre of oſcillation, SG the diſtance of the centre of gravity from the axis, and W the bodies' weight, the || ſum of the products $A \times AS^2 + B \times BS^2 + C \times CS^2$, &c. will $= SO \times SG \times W$: let therefore the weights of the bodies which conſtitute the ſyſtem be $\overset{\prime}{W}, \overset{\prime\prime}{W}, \overset{\prime\prime}{W}$; let the diſtances of the centres of oſcillation, belong- || Sect. VI. Prop. V. Cor. 3.

belonging to the bodies conſidered to revolve ſeparately round the axis, be $\overset{\prime}{O}, \overset{\prime\prime}{O}, \overset{\prime\prime\prime}{O}$, &c. reſpectively, and the diſtances of the centres of gravity be $\overset{\prime}{G}, \overset{\prime\prime}{G}, \overset{\prime\prime\prime}{G}$; then the force which accelerates $D = \dfrac{P \times SD^2}{P \times SD^2 + \overset{\prime}{O}\overset{\prime}{G}\overset{\prime}{W} + \overset{\prime\prime}{O}\overset{\prime\prime}{G}\overset{\prime\prime}{W} + \overset{\prime\prime\prime}{O}\overset{\prime\prime\prime}{G}\overset{\prime\prime\prime}{W}}$.

When this rule is applied to practice, it is manifeſt, that the axis of motion muſt paſs through the common centre of gravity, (this being the beſt method of rendering the moving force conſtant) in which caſe the centre of oſcillation of the whole ſyſtem is * infinitely diſtant; but both the centres of gravity and oſcillation of the ſeparate parts are determined as if theſe bodies revolved round the axis of motion ſeparately: thus, let AB repreſent a cylinder, at the extremity of which two equal ſpheres are fixed, ſo that the axis of the cylinder ſhall paſs through the centres of the ſpheres F and G. Let $SF = s$, $SG = a$, $FH = r$, the radius of the cylinder $= c$, and $SA = b$, and let the weight of either ſphere $= W'$, and that of the cylindrical arm $SA = \overset{\prime\prime}{W}$; alſo let $S\overset{\prime\prime}{G} = \dfrac{SA}{2}$, $\overset{\prime\prime}{G}$ being the centre of gravity of SA, then if the ſphere A were to revolve horizontally round the vertical axis LK the diſtance of the centre of oſcillation from the axis of motion $= a + \dfrac{2r^2}{5a}$, let this $= S\overset{\prime}{O}$; alſo, if the cylinder SA revolves round the axis paſſing through S, the diſtance of the centre of oſcillation from the axis $= \dfrac{4b^2 + 3c^2}{6b}$; let this be repreſented by $S\overset{\prime\prime}{O}$; then if a force P be applied to the point D, the force by which P is accelerated will be $\dfrac{P \times SD^2}{P \times SD^2 + 2S\overset{\prime}{O} \times S\overset{\prime}{G} \times \overset{\prime}{W} + 2S\overset{\prime\prime}{O} \times S\overset{\prime\prime}{G} \times \overset{\prime\prime}{W}}$, or if $SD = d$, the force $= \dfrac{P \times d^2}{Pd^2 + 2W \times \dfrac{5a^2 + 2r^2}{5} + 2\overset{\prime\prime}{W} \times \dfrac{4b^2 + 3c^2}{12}}$.

* Sect. VI. Prop. VI.

Fig. LIII.

Fig. LIV.

When a body revolving round an axis which paſſes through its centre of gravity, is contained under any ſingle geometrical figure, the force of acceleration will be at once obtained by finding the ſum of the products which ariſe from multiplying each particle into the ſquare of its diſtance from the axis ; thus let a cylinder DEB revolve round its axis, paſſing through the centre S by the action of a force P applied at D, the force which accelerates D will be $\frac{P\times SD^2}{A\times AS^2+B\times BS^2+C\times CS^2,\ \&c.}$: ſuppoſing A, B and C to be the particles which compoſe the cylinder ; all theſe particles may be referred to one plane, i. e. that of EBD. To find the ſum of $A\times AS^2+B\times BS^2$, &c. let $SA=x$, $SD=d$, the cylinder's weight $=W$, and let a circle AFH be drawn through A with the centre S and diſtance SA. Moreover, let a be a point contiguous to A, and with the centre S and diſtance Sa deſcribe a circle afh: the area intercepted between theſe two circles will be to the whole area, as the weight of an increment of the cylinder A is to its whole weight ; but if $p=3.14159$, the circumference of the annulus $=2px$, and its area $=2px\dot{x}$, and ſince the whole area $=pd^2$, we have this proportion $2px\dot{x} : pd^2 :: A : W$, or $A=\frac{2px\dot{x}\times W}{pd^2}$, and this particle multiplied into the ſquare of its diſtance from the axis $=\frac{2W\times x^3\dot{x}}{d^2}$, and the ſum of all the products ſimilarly taken $=\frac{Wx^4}{2d^2}$, or when $x=d$, this ſum $=\frac{Wd^2}{2}$; the moving force therefore is in this caſe P and the maſs moved $\frac{Wd^2}{2d^2}=\frac{W}{2}$. The force therefore which accelerates the point $D=\frac{2P}{W}$. If the force P ſhould conſiſt of a weight poſſeſſing inertia, applied to turn the cylinder by means of a line going round it, and terminating in P, the inertia of P muſt be added to the maſs moved, this will give the accelerating force $\frac{2Pd^2}{2Pd^2+Wd^2}=\frac{2P}{2P+W}$. In the ſame manner, let a ſolid be formed by the revolution of any plane figure round a ſtrait line conſidered as an axis, if the ſolid ſhould be put into motion round this axis by the action of any force applied at a given point, the force by which that point is accelerated may be determined.

By theſe propoſitions the revolving motion of regular bodies may be aſcertained from the neceſſary data, and of irregular bodies alſo when the axis of motion paſſes not through the centre of gravity; for if the diſtances of the centres of gravity and * oſcillation from the axis of motion can be found by experiment, the diſtance of the centre of gyration from the axis will be a mean † proportional between them. But when a ſyſtem of irregular bodies revolves round an axis which paſſes through their common centre of gravity, the ſyſtem muſt be cauſed to vibrate as a pendulum round ſome other axis parallel to the former: and if the diſtance of the centre of oſcillation from the axis of vibration be thence inferred, this with the diſtance of the centre of gravity from the centre of oſcillation will give the diſtance of the centre of gyration from the axis which paſſes through the centre of gravity, and round which the ſyſtem revolves.

* Prop. VI. Cor. 4.

† Sect. VI. Prop. VIII. Cor. 1.

XIII.

Let SIK repreſent the plane in which any irregular body or ſyſtem of bodies revolves round an axis of motion paſſing through the centre of gravity G: having given the weight of the ſyſtem and the point D, to which a given force is applied, it is required to aſſign the diſtance of the centre of gyration from the axis of motion.

Fig. LVI

Let G be the common centre of gravity of the ſyſtem, and ſuppoſe the whole to be referred to the plane *SIK*, let the ſyſtem revolve in this plane round a fixed horizontal axis which paſſes through *G*: through *G* draw any line *SGo*, and with the centre *G* and any diſtance *GD* draw the circle *SDIK*, and ſuppoſe all the matter which is ‡ coincident with the circumference *SIK* juſt deſcribed, to be collected into *A*: in like manner imagine all the matter contained in other concentric circles to be collected into the correſponding points in the line *Go*, let *g* be the

‡ Sect. VI. Prop. I. Cor. 6.

centre of gravity of the matter contained in this line, and let o be the centre of oſcillation; then if a weight $= \frac{W \times So \times Sg}{GD^2}$* were collected into the circumference SIK, or any point of it D, the other parts of the ſyſtem being removed, the point D would be accelerated in the ſame manner as when the ſyſtem itſelf is acted on by the ſame force applied at the ſame diſtance from the axis. In this caſe the force which accelerates †$D = \frac{SD^2 \times P}{Sg \times So \times W}$, or if the inertia of P is conſidered, the force $= \frac{SD^2 P}{P \times SD^2 + Sg \times So \times W}$: but ſince the figure of the ſyſtem is not geometrical, the quantity $W \times \frac{Sg \times So}{SD^2}$, or the equivalent weight cannot be eſtimated by theory; it may be determined by experiment in the following manner.

Let a weight P be applied to communicate motion to the ſyſtem by means of a very ſlender and flexible line going round the wheel SDK, through the centre of which the axis paſſes: let this weight deſcend from reſt through any convenient ſpace s inches, and let the obſerved time of its deſcent be t ſeconds, then if l be the ſpace through which bodies deſcend freely by gravity in one ſecond, the equivalent weight ſought $= \frac{W \times Sg \times So}{SD^2} = \frac{P \times t^2 l}{s} - P$. Let this weight $= Q$, and we ſhall obtain the ſum of each particle multiplied into the ſquare of its diſtance from the axis of motion; and conſequently the diſtance of the centre of gyration from the ſame axis: for ſince $Q : W :: Gg \times Go : GD^2$,† we have $A \times GA^2 + B \times GB^2 + C \times GC^2$, &c. $= GD^2 \times Q = Go \times Gg \times W$, and ſince $Go \times Gg = GR^2$‡, it follows, that the diſtance of the centre of gyration from the axis, or $GR = GD \times \sqrt{\frac{Q}{W}}$. This having been once determined, the force which accelerates any point of the ſyſtem in other caſes, at whatever diſtance the moving force be applied, will be known. For the angular velocity of the ſyſtem will always be the ſame, as if the whole maſs were § collected into the centre of gyration.

An experimental method is alſo delivered by Bernoulli, and long ago ‖ tranſcribed: a demonſtration of it may be here inſerted as a further illuſtration of the principles of rotation. In any irregular figure referred to the plane SIK, let G be the centre of gravity, D the point to which a moving force p is applied to turn the ſyſtem round an axis which is perpendicular to the plane SIK. With the centre G and diſtance GD deſcribe a circle SLM; then taking any point in the circumference S, let the ſyſtem vibrate in the plane SIK perpendicular to the axis which paſſes through S, and let O be the centre of oſcillation of the ſyſtem ſo vibrating: then let $GO = b$, $SG = GD = d$, the weight of the ſyſtem $= W$, and the moving force $= p$: the force which accelerates $D = \frac{p \times GS}{p \times GS + w \times GO} = \frac{pd}{pd + wb}$: the demonſtration of this rule follows immediately from the preceding principles; for ſince the centre of gravity is G, and of oſcillation O, it follows, that * $SG \times GO \times w = A \times AG^2 + B \times BG^2 + C \times GC^2$, or the ſum of all the products which are formed by multiplying each particle in the ſyſtem into the ſquare of its diſtances from the axis, when that axis paſſes through the common centre of gravity: wherefore $SG \times GO = bd = \frac{A \times AG^2 + B \times BG^2 + C \times CG^2}{w}$ = the ſquare of the diſtance of the † centre of gyration from the axis, when the ſyſtem revolves round G; let the maſs w be collected into the centre of gyration at the diſtance $\sqrt{bd}$ from the axis; then will the communication of motion be the ſame as if the equivalent weight $w \times \frac{bd}{d^2}$ were collected in the point D‡, the other parts of the ſyſtem being removed: here the force which communicates motion to the point D is p, the maſs moved $p + \frac{bw}{d}$, and the ‖ force which accelerates D

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* Sect. VI. Prop. VIII. Cor. 2.

† Sect. VI. Prop. VIII.

† Sect. VI. Prop. X. ad finem.

‡ Ibid.

§ Sect. VI. Prop. VIII.

‖ Petrop. Comm. Vol. II. p. 214.

$$= \frac{p}{p + \frac{bw}{d}} = \frac{pd}{pd + bw}.$$

The diſtance SO is obtained practically thus; ſuppoſe that the ſyſtem performed n leaſt vibrations in t ſeconds, if $l = 193$ inches, $p = 3.14159$, &c. $SO = \frac{2lt^2}{n^2p^2}$ the length required.

Fig. LVI.

* Sect. VI. Prop. VI. Cor. 3.

† Sect. VI. Prop. VIII.

‡ Sect. VI. Prop. VIII. Cor. 2.

‖ Sect. I. Prop. IX.

The objection againſt the application of this method to practice is the difficulty of aſcertaining the diſtance of the centre of oſcillation from the axis S, or the ſpace SO by experiment.

This ariſes from the friction of the axis, the air's reſiſtance and the inequality of the arcs through which the pendulum vibrates, to all which it is difficult to apply the exact corrections; and it is obvious, that ſince the lengths of pendulums vary in a duplicate ratio of the times of vibration, a ſmall error in eſtimating the time cauſes a conſiderable variation in the lengths deduced: ſo ſmall an error as 9 ſeconds in 10 hours cauſes * a change of length in a pendulum which vibrate ſeconds $= \frac{1}{41}$ part of an inch, and this error even ſome pendulum clocks are liable to: but when pendulums are ſuſpended in a manner leſs accurate, the danger of error in determining the centre of oſcillation becomes much greater, the lengths thus deduced having been miſtaken by perſons well verſed in experiments, but who have truſted to experiment only, by $\frac{1}{2}$, or even a greater proportion of the true length.

* Sect. IV. Prop. I. Cor. 8.

In the propoſitions concerning rotation which have been demonſtrated, the axis of motion has been ſuppoſed fixed; but there are other caſes in which the axis itſelf moves with the common centre of gravity through which it paſſes. To exemplify this, let G repreſent the centre of gravity of a ſphere placed and ſuſtained quieſcent on an inclined plane, by a force which acts in a direction parallel to the plane: it is known, that in this caſe, the force in the direction of the plane is to the body's weight, as the height of the plane to the length. When therefore the ſphere is left to itſelf, it will endeavour to deſcend along the plane by a force which is to its natural gravity as the plane's height is to its length.

Fig. LVII.

Part of the ſphere's natural weight is loſt by its preſſure againſt the plane in a direction perpendicular to it: let the line AB repreſent in quantity and direction the ſphere's natural gravity; this being reſolved into two, EB perpendicular to plane, and AE in the direction of it. Then the force which impels the ſphere will be that part of its natural weight which is expreſſed by the fraction $\frac{AE}{AB}$. If the ſphere was at liberty to move freely in the direction

tion of this force, each particle would partake of an equal motion, being the ſame as that with which the centre of gravity moved: and this would be the ſphere's real motion were there no adheſion between the ſurfaces which prevented the ſphere from ſliding down the plane; it is upon this principle only, that the laws demonſtrated by Galilæo can be applied to the deſent of bodies along inclined planes. Indeed the ratios of the ſpaces deſcribed on a given inclined plane, will be the ſame in given circumſtances, whether the ſphere rolls or ſlides; but the abſolute quantity of them cannot be from hence aſſigned: and as the ſurfaces cannot be made ſo ſmooth as to prevent their rotation when they deſcend along inclined planes, it muſt not be expected that their motion will correſpond with the theory, unleſs the rotation be taken into account. To conſider this a little further: While the impelling force acts on the centre of gravity *G*, in the direction parallel to the plane *HG*, the point *A*, which is contiguous to the plane's ſurface, will be prevented from moving along with the centre of gravity by the adheſive force exiſting between the point *A* and the plane: by which means, while the centre of gravity deſcends along the plane in a right line, the ſphere will revolve on an axis, and all the particles except thoſe in the axis of motion will deſcribe cycloids.

The force which accelerates the centre of gravity of a ſphere or cylinder, when rolling down an inclined plane, is next to be conſidered.

XIV.

The force which accelerates the centre of gravity of a ſphere, while it rolls down an inclined plane, is to the force by which it would be accelerated were it to ſlide, in the ratio of five to ſeven.

Let the height of the plane be *H*, its length *L*, and the ſphere's weight $= W$; then will the force by which the centre of gravity endeavours to deſcend along the plane be $\frac{W \times H}{L}$. Alſo, if the ſphere were to ſlide, the weight moved being $= W$, the accelerating force $= \frac{H}{L}$: but ſince the ſurface revolves with a velocity equal to that with which the centre of gravity moves, the inertia ariſing from the ſphere will conſiſt not only of the ſphere's maſs of matter, but of the reſiſtance which proceeds from the rotatory motion. Let the ſphere G be ſuppoſed to deſcend by ſliding, then will the inertia be W; but let a line paſſing through the centre G be cauſed to communicate with the vertical circumference of a great circle of another ſphere H, equal in all reſpects to G, and moveable round an horizontal axis of motion by means of this ſtring: here it is manifeſt, that the circumference of the ſphere muſt be accelerated exactly in the ſame manner as the centre of gravity G. Now it is immaterial whether this rotatory motion be produced in the deſcending ſphere G, or in any other equal to it in weight and magnitude, the motion produced in a given time being the ſame in both caſes: it follows, that the force which accelerates the rolling ſphere will be the ſame as that which accelerates the ſphere which ſlides, and at the ſame time communicates a rotatory motion to an equal ſphere H, ſo that the circumference of the latter ſhall move equally faſt with the centre of the deſcending ſphere.

Let R be the centre of gyration of either ſphere, W the weight; then will the inertia of the ſphere H which reſiſts the communication of motion to its circumference to which the moving force is applied be the ſame, as if $W \times \frac{HR^2}{HD^2}$ [*] were collected into the circumference, ſince it moves equally faſt with the centre G; when therefore the ſphere rolls down the plane, the whole maſs moved or reſiſtance ariſing from inertia will be $W + \frac{W \times HR^2}{HD^2}$, or ſince in the ſphere $HR^2 = \frac{2HD^2}{5}$, the inertia will be $\frac{7W}{5}$, and becauſe the moving force is $\frac{W \times H}{L}$, it follows, that the force which accelerates the centre of gravity $G = \frac{5W}{7L}$, which

[*] Sect. VI. Prop. VIII. Cor. 2.

which is to the force which would accelerate the ſphere when ſliding down as 5 : 7.

Cor. 1. The abſolute force whereby motion is generated in the circumference of the ſphere A is $= \frac{2W}{7} \times \frac{H}{L}$. For the equivalent weight at the circumference of the the ſphere H is $= \frac{2W}{5}$, and the force which accelerates the circumference $= \frac{5}{7} \times \frac{H}{L}$, wherefore the moving force which is always ‡equal to the weight moved multiplied into the number expreſſing the accelerating force will be $= \frac{2W}{5} \times \frac{5}{7} \times \frac{H}{L} = \frac{2W}{7} \times \frac{H}{L}$. ‡ Sect. I. Prop. IX.

Cor. 2. In the ſame manner, let a cylinder revolve down an inclined plane the axis being always horizontal, the force which accelerates the axis will be found that part of gravity which is expreſſed by the fraction $\frac{2}{3} \times \frac{H}{L}$.

Let ABC repreſent the ſection of any irregular figure whatever paſſing through its centre of gravity G: with the centre G and any diſtance GS deſcribe a circle. Suppoſe the whole ſyſtem to roll down an inclined plane, being ſuſtained on the circumference DS which is always in contact with the plane and vertical. If R be the centre of gyration of the figure, the force which accelerates the centre of gravity will be that part of the acceleration of gravity, which is expreſſed by the fraction $\frac{GD^2}{GD^2 + GR^2} \times \frac{H}{L}$, and that part of the moving force which cauſes the rotation $= \frac{W \times GR^2}{GD^2 + GR^2} \times \frac{H}{L}$, the demonſtration of which follows from the preceding ſolution. Fig. LVIII.

Cor. 3. Let the figure be ſuſpended on an axis paſſing through any point S in the circumference; and when it vibrates in its own plane, let O be the centre of oſcillation: the force which accelerates the centre G will alſo $= \frac{SG}{SO} \times \frac{H}{L}$: for ||$GR^2 = \frac{A \times AG^2 + B \times BG^2 + C \times CG^2, \text{ \&c.}}{A + B + C} =$ || Sect. VI. Prop. VI. Cor. 3.

$GO \times SG$, or $GO \times GD$; ſubſtituting therefore $GO \times GD$ for GR^2 in the former expreſſion, we have $\frac{GD^2}{GD^2 + GR^2} = \frac{GD^2}{GD^2 + GO \times GD} = \frac{GD}{GD + GO} = \frac{SG}{SO}$, and the force which accelerates the centre of gravity $= \frac{SG}{SO} \times \frac{H}{L}$, and that part of the moving force by which the rotation of the circumference is cauſed $= \frac{GO \times W}{SO} \times \frac{H}{L}$.

The force whereby ſpheres, cylinders, &c. are cauſed to revolve as they move down an inclined plane inſtead of ſliding, is the adheſion of their ſurfaces occaſioned by the preſſure of the rolling body againſt the plane: this preſſure is part of the body's weight; for the weight being reſolved into two, one force in the direction of the plane, and the other perpendicular to it, the latter is the force of the preſſure, and while the ſame body rolls down the plane, will be that part of its weight which is expreſſed by the coſine of the angle of the plane's elevation divided by radius. It appears from hence, that ſince the coſine of an arc is decreaſed while the arc increaſes, the preſſure may by elevating the plane to a certain angle above the horizon become ſo ſmall, that the adheſion ſhall be leſs than the force neceſſary to make the circumference of the ſphere revolve ſo faſt as the centre of gravity deſcends: in this caſe the ſphere deſcends by ſliding partly and partly by rotation.

The ſame will happen when the ſurfaces of the revolving body and of the plane are ſo ſmooth, that their mutual adheſion is leſs than the moving force or weight neceſſary to generate the rotation of the ſphere; that is, referring to the example juſt ‖ mentioned leſs than $\frac{W \times GO}{SO}$. But a body may be cauſed to roll at all elevations of the inclined plane, by winding a line round the cylindrical ſurface in a manner repreſented in fig. LVIII. as the cylinder G deſcends it unwinds itſelf from the line, which in this caſe ſupplies the place of the adheſion between the ſurfaces; let the rolling body be ſuſpended on any point of the circumference S, let G be its centre of gravity being coincident with the axis of the cylinder, and ſuppoſe O to be the centre of oſcillation of the figure, when vibrating round S in its own plane, then will the force which

‖ Supra.

which accelerates the centre of gravity be that part of the accelerating force of gravity which is expreſſed by $\frac{SG}{SO} \times \frac{H}{L}$, and if the body's weight $= W$, the abſolute force or weight whereby the rotation is || cauſed $= W \times \frac{GO}{SO} \times \frac{H}{L}$; wherefore let a line KS be drawn coincident with the plane, and a wheel K be placed in a vertical plane in contact with KS; moreover, let a line wound round the deſcending cylinder DS be applied over the wheel or pully, and to the extremity of this line, let a weight $p = W \times \frac{GO}{SO} \times \frac{H}{L}$ be affixed; this weight will remain quieſcent while the body ABC rolls down the plane, being juſt equal to the tenſion of the ſtring Kp.

|| Sect. I. Prop. IX.

It is obvious, that if the plane is elevated ſo as to be perpendicular to the horizon, the body ABC will deſcend in a vertical line, by the action of an accelerating force $= \frac{SG}{SO}$, ſince $H = L$, and the weight p, which is equivalent to the tenſion of the ſtring Kp, will become $= W \times \frac{GO}{SO}$.

XV.

Fig. LIX.

Let ABC repreſent a body moveable round its centre of gravity S, through which an horizontal axis of motion paſſes; let R repreſent the centre of gyration when the whole maſs is collected into the line SGRO, each particle keeping its reſpective diſtance from the axis: having given the weight of the whole body, and of P which communicates motion to the ſyſtem by means of a line DP wound round the circle DEF, it is required to aſcertain the angular velocity generated in the ſyſtem in a given time.

Let the diſtance of the centre of gyration from the axis $SR = r$, $SD = d$, the weight of the body $ABC = w$, then will the force which * accelerates D be $\frac{pd^2}{r^2w}$, were p deſtitute of inertia; but if the inertia of p be conſidered, the force of acceleration is $\frac{pd^2}{d^2p + r^2w}$: let $c = 3.14159$, $l = 193$ inches, then in the time t the velocity generated in the point D, or deſcending weight p, will be that of $\frac{2ltpd^2}{d^2p + r^2w}$ † inches in a ſecond; and ſince the circumference of the circle $EFD = 2cd$, the angular velocity generated in the time $t = \frac{360^\circ \times 2ltpd^2}{2cd \times \overline{d^2p + r^2w}} = \frac{360^\circ ltpd}{c \times \overline{d^2p + cr^2w}}$ degrees, or $\frac{ltpd}{cd^2p + cr^2w}$ revolutions in a ſecond.

Cor. 1. The angular velocity generated in the ſyſtem during the deſcent of the weight p through any ſpace $s = \sqrt{\frac{lsp}{c^2d\,p + c^2r^2w}} \times 360^\circ$ degrees, or $\sqrt{\frac{lsp}{c^2d^2p + c^2r^2w}}$ revolutions in a ſecond. Let F be the force which accelerates p; then the abſolute velocity generated in D during the deſcent of p through a ſpace s by the accelerating force $F = \sqrt{4lsF}$ ‡ inches in a ſecond; and any arc $\sqrt{4lsF}$ is to $2cd$ the circumference of the circle EFD, as A° the angle ſubtended by $\sqrt{4lsF}$ is to 360°: wherefore $A^\circ = \sqrt{4lsF} \times \frac{360}{2cd}$, being the angular velocity expreſſed in the number of degrees which would be deſcribed in a ſecond by the ſyſtem, were the acceleration to ceaſe immediately after the weight p has deſcended through the ſpace s; and reſtoring the value of $F =$

pa^2

* Sect. VI. Prop. VIII. Cor. 2.

† Sect. III. Prop. II.

‡ Sect. III. Prop. V. Cor. 3.

$\frac{pd^2}{d^2p+r^2w}$, the angular velocity $= \sqrt{\frac{lsp}{c^2d^2p+c^2r^2w}}$ × 360° degrees, or $\sqrt{\frac{lsp}{c^2d^2p+c^2r^2w}}$ revolutions in a ſecond.

Cor. 2. The ſpace deſcribed by the weight p in its deſcent from reſt during t * ſeconds $= \frac{lt^2pd^2}{d^2p+r^2w}$, and conſequently the time of deſcribing any ſpace s, that is, $t = \sqrt{\frac{sd^2p+sr^2w}{lpd^2}}$ ſeconds.

* Sect. III. Prop. IV. Cor. 5.

Cor. 3. The ſpace deſcribed by p from reſt while an angular velocity of n revolutions in a ſecond is generated $= \frac{n^2c^2d^2p+n^2c^2r^2w}{lp}$.

Cor. 4. The force which ‡ accelerates the centre of gyration $= \frac{pdr}{d^2p+r^2w}$.

‡ Sect. VI. Prop. II.

Cor. 5. The abſolute velocity generated in the weight p while it deſcends from reſt through the ſpace $s = \sqrt{\frac{4lspd^2}{d^2p+r^2w}}$, and the velocity generated during the ſame deſcent in the centre of gyration $= \sqrt{\frac{4lspr^2}{d^2p+r^2w}}$.

Cor. 6. The velocity generated in the centre of gyration in the time $t = \frac{2ltdpr}{d^2p+r^2w}$ inches in a ſecond.

While motion is communicated to a ſyſtem of revolving bodies by the action of a deſcending weight p, the force of this weight is employed partly to move the ſyſtem and partly to move itſelf: but it frequently happens in the application of the principles of rotation to practice, that the inertia of the weight p is too inconſiderable to have ſenſible effect.

XVI.

In any ſyſtems of bodies revolving by the action of weights (conſidered as without inertia) the abſolute forces which impel the centres of gyration, are in a compound ratio of the quantities of matter or weights of the ſyſtems and the velocities generated in the centres of gyration, if the times of motion be equal: and the abſolute forces or weights which are applied to communicate motion to the ſyſtems, are in a compound ratio of the quantities of matter moved or weights of the ſyſtems and a duplicate ratio of the velocities generated in the centres of gyration, provided the ſpaces deſcribed by the deſcending weights be equal.

The notation of the laſt propoſition remaining, let v be the velocity generated in the centre of gyration in t ſeconds, then ‡ we have $v = \frac{2ltpdr}{d^2p + r^2w}$, and when the inertia of $p = 0$, $v = \frac{2ltpd}{wr}$, and $\frac{pd}{r} = \frac{v \times w}{2lt}$; but the abſolute force which acts on D is p, and that which acts on the centre of gyration $= \frac{pd}{r}$: wherefore the force which acts on the centre of gyration $\frac{pd}{r}$ is proportional to to the quantity $\frac{v \times w}{2lt}$, and in a given time $2lt$ being con-

ſtant,

‡ Prop. XV. Cor. 6.

ſtant, will be as $v \times w$, that is, as the velocity generated into the quantity of matter, which is the firſt part of the propoſition.

The velocity generated in the centre of gyration, while the deſcending weight p deſcribes the * ſpace s is $\sqrt{\frac{4lsp}{w}}$ inches in a ſecond: let this $= v$, we have therefore $v^2 = \frac{4lsp}{w}$ and $p = \frac{v^2 w}{4ls}$; but p is the abſolute force which acts on the ſyſtem, and this force or weight, if the ſpace s which it deſcends through be the ſame, will be as $v^2 \times w$, or in the ratio of the quantities of matter moved and a duplicate ratio of the velocities generated jointly, which is the ſecond part of the propoſition to be proved.

* Sect. VI. Prop. XV. Cor. 5.

The following concluſions are inferred from the two laſt propoſitions, on a ſuppoſition that the inertia of the moving force is $= 0$.

1. If a force p communicating motion to a given ſyſtem of revolving bodies, is applied at different diſtances SD, SI from the axis, the forces which accelerate the points D and I in the two caſes, are in a direct duplicate ratio of theſe diſtances.

2. The notation remaining the ſpace deſcribed by the weight p from reſt in t ſeconds $= \frac{lt^2 d^2 p}{wr^2}$.

3. The abſolute velocity generated in the deſcending weight p in deſcribing a ſpace s from reſt $= \sqrt{\frac{4lsd^2p}{wr^2}}$.

4. The velocity generated in p during its deſcent from reſt in t ſeconds $= \frac{2ltd^2p}{wr^2}$ inches in a ſecond.

5. The angular velocity generated in the ſyſtem in t ſeconds $= \frac{ltpd}{cr^2w} \times 360$ degrees, or $\frac{ltpd}{cr^2w}$ revolutions in a ſecond.

6. The angular velocity generated during the deſcent of the weight p through a ſpace $s = \sqrt{\frac{lsp}{c^2r^2w}}$ revolutions in a ſecond.

7. The velocity generated in the centre of gyration in t ſeconds $= \frac{2ltdp}{wr}$.

8. The velocity generated in the centre of gyration during the deſcent of the weight p through a ſpace $s =$ $\sqrt{\frac{4lsp}{w}}$.

9. The time in which the centre of gyration deſcribes from reſt any ſpace $s = \sqrt{\frac{swr}{pld}}$.

10. The velocity generated in the centre of gyration during the time that point deſcribes any ſpace $S =$ $\sqrt{\frac{4lSpd}{wr}}$.

From the 8th Cor. it follows, that the motion of the centre of gyration generated by the force of a weight deſcending through a given ſpace S, applied at any diſtance from the axis whatever, is the ſame as would be generated in the centre of gravity of the whole ſyſtem, when at liberty to move freely; if the given weight were applied to accelerate it from reſt through a ſpace $= S$.

From theſe propoſitions the principle denominated Conſervatio virium vivarum, as far as it is conſiſtent with truth, is readily deduced. This principle was formerly ſuppoſed to conſtitute a theory not entirely conſiſtent with that which is eſtabliſhed upon the Newtonian laws of motion; but the different opinions concerning the meaſures of force having been long acknowledged a diſpute about words, it has become unneceſſary to conſider the ſubject in a controverſial manner.

According to this theory, Leibnitz and Bernoulli eſtimate the force of bodies in motion by the ſum of the products, which are formed by multiplying each particle into the ſquare of its velocity.

Though this has long ago been demonſtrated by various writers to be falſe, when applied in a general way to the colliſion of bodies; yet it is certainly true as far as regards the communication of motion by conſtant acceleration, under certain reſtrictions and conditions, and is in fact no other than a propoſition eaſily deduced from the Newtonian laws of motion. This has been ſufficiently ſhewn when the motion communicated is *rectilinear; the application * Sect. III. Prop. VI. of it to bodies which revolve round a fixed axis will ſerve for the further illuſtration of the theory.

XVII.

Let A B C repreſent the plane in which a Fig. LIX. body or ſyſtem of bodies revolves round an horizontal axis which paſſes through the centre of gravity S. Suppoſe motion to be communicated to this ſyſtem by the action of a weight P, affixed to a line wound round the circumference of a circle, the plane of which is the ſame with that of the rotation, and the centre coincident with the axis of motion: then if P by deſcending through any ſpace S, generates velocity in the revolving ſyſtem, the ſum of the products of each particle in the ſyſtem, multiplied into the ſquare of the velocity generated in it, will be equal to the ſum of the products of each particle in the body P into the ſquare of the velocity which would be generated in it by gravity, if it deſcended freely from reſt through the ſame ſpace S.

The notation of the preceding propoſitions remaining, the velocity generated in the point D during the deſcent of p|| through the ſpace $s = \sqrt{\frac{4lspd^2}{wr^2}}$, and || Sect. VI. Prop. XVI. Cor. 3.

conſequently the velocity of any particle $A = \sqrt{\frac{4lspd^2}{wr^2}} \times \frac{SA}{d}$, and the ſquare of the velocity generated in $A = \frac{4lspd^2}{wr^2} \times \frac{SA^2}{d^2}$: and in like manner the ſquare of the velocity generated in any other particle B during the deſcent of p through s will be $\frac{4lspd^2}{wr^2} \times \frac{SB^2}{d^2}$, and the ſum of the products of each particle into the ſquare of its velocity $= \frac{4lspd^2}{wr^2} \times \frac{A \times AS^2 + B \times BS^2 + C \times CS^2,\ \&c.}{d^2}$: but $A \times AS^2 + B \times BS^2$ ‡ $+ C \times CS^2$, &c. $= wr^2$, wherefore the ſum of the products before mentioned $= 4lsp$. Now let the weight p deſcend from reſt through the ſpace s by its natural gravity, the velocity generated ‖will be $\sqrt{4ls}$, and the ſquare of the velocity $= 4ls$, and this multiplied into each particle of the body $p = 4lsp$; the ſame as the ſum of the products of each particle in the ſyſtem into the ſquare of its velocity.

‡ Sect. VI. Prop. VIII. Cor. 1.

‖ Sect. III. Prop. V. Cor. 3.

The explanation of theſe propoſitions was the ſubject of a diſcourſe on the force of bodies in motion, delivered at a public courſe of lectures read at the univerſity of Cambridge, in the years 1776 and 1777; and the truth of them was confirmed by many experiments, as well on bodies which moved in right lines, as on thoſe which revolved round a fixed axis of motion: it was ſhewn, that the Newtonian meaſure of the quantities of motion by the joint ratios of the quantities of matter and velocities generated, and that of Leibnitz and Bernoulli, according to whom the forces communicated to bodies are proportional to the quantities of matter moved, and the ſquares of the velocities generated in them, are conſiſtent with each other, and with the theory of motion in general: and for the following reaſons: namely, ſince if the times of motion be the ſame, the forces which * generate velocities in bodies, are as the quantities of matter and velocities jointly, according to the Newtonian meaſure; whereas if the ſpaces deſcribed from reſt are the ſame, the forces to create motion in the bodies moved are as the quantities of matter and the † ſquares of the velocities jointly, as Bernoulli and others affirm in general, and without the neceſſary

* Sect. III. Prop. VII.

limitation: and it was obſerved, that the principle upon which the coincidence of the two theories depends, is the following propoſition, i. e. that when bodies are uniformly accelerated through equal ſpaces, the accelerating forces are as the ſquares of the velocities generated.

XVIII.

Let AB repreſent a ſtraight lever moveable round an horizontal axis of motion which paſſes through S: let the arms be SB, SA: ſuppoſe a weight W to be affixed to the extremity of the ſhorter arm, and to be raiſed by the weight P, applied at the extremity of the longer arm, when the lever is horizontal; and let it be required to aſſign in what time, W will be raiſed through a given altitude; the weight and inertia of the lever itſelf not being conſidered. Fig. LX.

When there is an equilibrium on any mechanic power, the proportion of the weight ſuſtained to the power ſuſtaining it, will in all caſes be aſſigned from having given the dimenſions of the mechanic power.

An equilibrium having been once formed, the ſmalleſt addition of weight will cauſe the body to which it is applied on either ſide to preponderate: in this caſe a certain degree of motion is generated, and ſince the uſes of the mechanic powers are not only to ſuſtain forces in equilibrio, but to raiſe weights and overcome reſiſtances, it is a problem of principal conſequence, to aſſign the abſolute quantity of motion generated by a known moving force, in given circumſtances.

Let $SA = a$, $SB = r$. Firſt to conſider the effects of the force p to turn the ſyſtem at the firſt inſtant of motion.

The force applied to B is the weight p, but ſince the weight w being ſuſpended from the extremity of the arm SA will balance a weight $= \frac{wa}{r}$ when placed at B, this force muſt be ſubducted from p; the impelling force therefore which acts on B upon the whole will be $p - \frac{wa}{r} = \frac{rp - wa}{r}$. The * inertia or maſs moved at B will be obtained by ſuppoſing the bodies p and w to be removed, and an equivalent maſs $p + \frac{wa^2}{r^2}$ to be concentrated in B; the inertia therefore at B will be that of the maſs $\frac{pr^2 + wa^2}{r^2}$, and the force which accelerates B or p at the firſt inſtant of motion when the lever is horizontal $= \frac{r^2p - raw}{pr^2 + wa^2}$‖. Let this $= F$; ſuppoſe the point p to have deſcended through the arc BP, and through P draw PE perpendicular to SB, and let $PE = x$, $SB = r$, then will the force which accelerates the point P be leſs than when the lever is horizontal, in the proportion of $SB : SE$, or of $r : \sqrt{r^2 - x^2}$, and will therefore $= F \times \frac{\sqrt{r^2 - x^2}}{r}$: let z be the altitude through which a body muſt fall freely to acquire the velocity of p, during the time of its deſcribing the elementary arc Po; and ſince the arc $Po = \frac{r \times \dot{x}}{\sqrt{r^2 - x^2}}$, the principles of acceleration † give us $\dot{z} = \frac{F \times \sqrt{r^2 - x^2}}{r} \times \frac{r\dot{x}}{\sqrt{r^2 - x^2}} = \dot{x}F$, wherefore $z = xF$, and if $l = 193$ inches, the velocity of p while it deſcribes $oP = \sqrt{4lFx}$ inches in a ſecond.

If $\dot{T}$ be put to repreſent the time of deſcribing the arc Po or $\frac{r\dot{x}}{\sqrt{r^2 - x^2}}$, this will give ‡ $\dot{T} = \frac{r\dot{x}}{\sqrt{r^2 - x^2}} \times \frac{1}{\sqrt{4lFx}} = \overline{\frac{r^2}{4lF}} \times \frac{\dot{x}}{\sqrt{r^2x - x^3}}$, the fluent of which

* Sect. VI. Prop. I. Cor. 3.

‖ Sect. VI. Prop. I. Cor. 3.

† Sect. III. Prop. V. Cor. 4.

‡ Sect. III. Prop. III.

which will be the time of deſcribing the arc Bp: this may be found by an infinite ſeries. Since $\frac{\dot{x}}{\sqrt{r^2x - x^3}} = \frac{\dot{x}}{r\sqrt{x}} + \frac{x^{\frac{3}{2}}\dot{x}.1}{r^3.1.2} + \frac{x^{\frac{7}{2}}\dot{x}.1.3}{r^5.2.4} + \frac{x^{\frac{11}{2}}\dot{x}\,1.3.5}{r^7\,2.4.6}$, &c. the fluent of $\frac{\dot{x}}{\sqrt{r^2x - x^3}}$ will $= \frac{2x^{\frac{1}{2}}}{r} \times \overline{1 + \frac{x^2 \times 1}{r^2.2.5} + \frac{x^4.1.3}{r^4\,2.4.9} + \frac{x^6.1.3.5}{r^6.2.4.6.13}}$, &c. and the time in which the weight p deſcribes the arc $Bp = \sqrt{\frac{x}{lF}} \times \overline{1 + \frac{x^2}{r^2.2.5} + \frac{x^4.1.3}{r^4.2.4.9} + \frac{x^6.1.3.5}{r^6.2.4.6.11}}$, &c. expreſſed in ſeconds. As the arms of a lever in raiſing weights are ſuppoſed to move through but ſmall angles, a few terms of the ſeries will be a ſufficient approximation: if the angle through which the weight w is raiſed, be not greater than thirty degrees, two terms will give the time true within about $\frac{1}{380}$ part of the whole.

In this caſe, the weight and figure of the materials of which the lever is compoſed, have not been taken into account. The following is a more general ſolution; let AB be the lever, w the weight moved by the power p, each acting in a direction perpendicular to the horizon. Let G be the common centre of gravity of the whole ſyſtem, including the weights p and w, and the lever itſelf, and O the centre of oſcillation when AB vibrates round the axis S: the force which accelerates B when the lever is * horizontal $= \frac{SG \times SB}{SG \times SO} = \frac{SB}{SO}$. If this be put $= F$, the time wherein p deſcends through a perpendicular ſpace x, and conſequently wherein w aſcends through $x \times \frac{SA}{SB} = \sqrt{\frac{x}{lF}} \times \overline{1 + \frac{x^2}{r^2\,2.5}}$, &c. $= \sqrt{\frac{x}{l} \times \frac{SO}{SB}} \times \overline{1 + \frac{x^2}{r^2.2.5}}$, &c.

* Sect. VI. Prop. V. Prop. VIII. and Prop. II.

Fig. LXI.

When a wheel and axle is employed to raiſe any weight q applied to the circumference of the axle, by means of the power p applied to the circumference of the wheel, the axis of the wheel is ſuppoſed horizontal, and the moving force being conſtant, the force which accelerates any given point in the ſyſtem will be obtained from the principles already demonſtrated, and conſequently the ſpace deſcribed by the elevated weight in a given time will be aſcertained.

XIX.

Fig. LXI. Let A B C repreſent a wheel and axle, its weight w, and let the axis be horizontal: having given a weight q applied to the circumference of the axle, and p applied to the circumference of the wheel in order to raiſe q, it is required to aſſign the ſpace deſcribed by the elevated weight q from reſt in any given time.

The abſolute force which impels D is p, and ſince q acts in a direction contrary to p with a force $= \frac{q \times SA}{SD}$, this being ſubducted from p will give $p - \frac{q \times SA}{SD} = \frac{p \times SD - q \times SA}{SD}$ for the force which upon the whole impels D. Let the centre of gyration of the wheel be R, then ſuppoſe the maſs of matter in the whole ſyſtem removed, if the maſs $\frac{w \times SR^2 + {}^{*}q \times SA^2 + p \times SD^2}{SD^2}$ be concentrated in D, the point D will be accelerated in the ſame manner as when the parts of the ſyſtem are diſpoſed as they are deſcribed in the problem. Since then the force which impels $D = \frac{p \times SD - q \times SA}{SD}$, and the

S. [illegible]. VI. [illegible] VIII. [illegible] 1.

inertia which reſiſts the communication of motion to D is $\frac{w \times SR^2 + q \times SA^2 + p \times SD^2}{DS^2}$, we have the force which accelerates $D = \frac{\overline{p \times SD - q \times SA} \times SD}{w \times SR^2 + p \times SD^2 + q \times SA^2}$, and conſequently the force which accelerates the weight w in its aſcent* $= \frac{\overline{p \times SD - q \times SA} \times SA}{w \times SR^2 + p \times SD^2 + q \times SA^2}$: let this $= F$, and $l = 193$ inches, then the ſpace deſcribed from † reſt by q in the time t ſeconds $= t^2 l F$ inches.

* Sect. VI. Prop. II.

† Sect. III. Prop. IV. Cor. 5.

Cor. 1. Let SD be to SA as $n : 1$; then if $SA = a$, $SD = na$: let $SR = ra$, [illegible] the force which accelerates the point $D = \frac{\overline{n^2 p - nq}}{wr^2 + pn^2 + q}$.

Cor. 2. If the inertia of the wheel and axle is evaneſcent or too ſmall to have ſenſible effect, the force which accelerates $D = \frac{n^2 p - nq}{pn^2 + q}$.

Cor. 3. If the inertia of the moving force is alſo $= 0$, the force which accelerates the point $D = \frac{n^2 p - nq}{q}$.

Cor. 4. If the maſs moved has no weight but poſſeſſes inertia only, the force which accelerates $D = \frac{n^2 p}{pn^2 + q}$.

Whenever motion is communicated to a body, a certain reſiſtance muſt have been overcome by the moving force: this reſiſtance is of various kinds, i. e. 1. The inertia of the maſs moved whereby it endeavours to perſevere in its ſtate of quieſcence, or of uniform motion in a right line. 2. That of a weight or other abſolute force oppoſed to the action of the moving power. 3. Obſtacles upon which the moving body impinging is retarded in its progreſs: ſuch for example is the reſiſtance which ariſes from the particles of a fluid through which a body moves. The eſtimation of theſe reſiſtances, and their effects in retarding the motion of bodies acted on by a given force, are deducible from the laws of motion, and conſtitute a part of the ſolution of almoſt all problems relating to the motion of bodies.

The moving forces alſo are of various kinds, i. e. the earth's gravity, muſcular power, the impact of bodies ſolid or fluid, &c. It has been ſhewn, that the effects of theſe moving forces which are exerted on bodies in order to create motion, excluſive of the reſiſtance oppoſed to them, depend on the various circumſtances of the time in which they act, and on the ſpaces through which the bodies moved are impelled, &c.

Theſe conſiderations are urged to ſhew, that from the great variety of undetermined conditions which may enter into mechanical problems, there muſt be of courſe various methods of producing the ſame mechanical effect; and it is a very material part of the art, conſidered either in a theoretical or practical view, to proportion the means to the end, and to effect this with all the advantages which the nature of the caſe is capable of. It is the due obſervation of theſe particulars which contributes to render mechanic inſtruments compleat, and the neglect of them defective in their conſtruction. This proper choice of means to produce mechanical effect is frequently the reſult of long continued experience independent of all theory; the knowledge of which however when immediately applied to practice would ſave the artiſt much time and trouble, as well as would be productive of other advantages which experience alone muſt be deſtitute of.

XX.

Fig. LXI. ABC is a wheel and axle moveable round an horizontal axis which paſſes through s. Suppoſe a given weight q, which is applied to the circumference of the axle, to be raiſed by the application of a given moving force p, which is applied to the circumference of the wheel: let it be required to aſſign the proportion of the radii of the wheel and axle, ſo that the time in which the weight w aſcends through any given ſpace ſhall be the leaſt poſſible.

Let the given radius of the axle $SA = a$, the radius of the wheel ſought $= x$, let the wheel's weight $= w$, the diſtance of the centre of gyration from the axis of motion $= r$, then will the force which accelerates the weight p during its *deſcent $= \frac{x^2p - qax}{wr^2 + px^2 + qa^2}$, that which accelerates q in its † aſcent $\frac{axp - qa^2}{wr^2 + px^2 + qa^2}$.

The ſquare of ‡time therefore in which any ſpace s is deſcribed by the aſcending weight $q = \frac{s}{l} \times \overline{\frac{wr^2 + px^2 + qa^2}{apx - qa^2}}$, which is to be the leaſt poſſible by the problem; making therefore its fluxion $= 0$, we have $p^2ax^2\dot{x} - 2pqa^2x\dot{x} - wr^2pa\dot{x} - pqa^3\dot{x} = 0$, which being reduced gives $x = \frac{aq + \sqrt{q^2a^2 + pwr^2 + qpa^2}}{p}$.

Cor. 1. Suppoſe the inertia of the wheel to be evaneſcent, then $x = \frac{aq + \sqrt{q^2a^2 + qpa^2}}{p}$, and if $q = p$, that is, the weight moved being equal to the moving force, it will follow, that $x = a + a \times \sqrt{2} = a \times \overline{1 + \sqrt{2}}$.

Let ABC repreſent a cylindrical wheel, the radius of which $= 10$ inches, and its weight 20 oz. let the radius of the axle $SA = 1$ inch, the weight to be raiſed through any given ſpace be 100 oz. the moving force by which it is raiſed $= 33$ oz. ſince the radius of the cylinder $= 10$ inches, the diſtance of the centre of gyration from the axis $= \sqrt{50}$ inches ‖: then to find a diſtance SD, at which the moving force p is applied, ſo that the time in which q aſcends through a given ſpace ſhall be the leaſt poſſible, we have $q = 100$, $p = 33$, $w = 20$, $r^2 = 50$, $a = 1$, and the diſtance ſought $= \frac{100 + \sqrt{10000 + 33000 + 3300}}{33}$ $= \frac{315.17}{33} = 9.55$ inches.

This

* Sect. VI. Prop. XIX.

† Sect. VI. Prop. II.

‡ Sect. III. Prop. IV. Cor. 5.

‖ Sect. VI. Prop. XII. p. 224.

This therefore will be the moſt convenient diſtance to apply the given moving force, when the chief object is to leſſen the time of aſcent. If it be required to aſſign the diſtance *SD*, when the moment communicated to w while it aſcends through a given ſpace is the greateſt poſſible, the ſolution will be the ſame as before, which therefore anſwers to two conditions; that is, it will render the time in which q aſcends through a given ſpace the leaſt, and the moment generated during the ſame aſcent the greateſt poſſible.

Fig. LXII. If the weight q inſtead of aſcending in a vertical direction is drawn along an horizontal plane, and the ſurfaces be perfectly free from friction, the weight of q will $= 0$. In this caſe if it be required to aſſign the diſtance *SD*, at which if the given force p be applied, the time of deſcribing a given ſpace ſhall be the leaſt, and the moment generated in q the greateſt poſſible, we ſhall have $SD = \sqrt{\frac{r^2 w + q a^2}{p}}$, and if the inertia of the wheel ſhould be too ſmall to have ſenſible effect, $SD = a \times \sqrt{\frac{q}{p}}$.

Thus, let the quantity of matter to be drawn along the plane be four times greater than that which is contained in the moving force, the radius of the axle *SA* being given; in order that it may be impelled with the greateſt velocity poſſible and with the greateſt moment, the radius of the wheel ſhould be double to that of the axle, when the inertia of the wheel is not conſidered.

XXI.

Fig. LXI. ABC is a wheel and axle, the axis of which is horizontal: having given a moving force or weight p acting on the circumference of the wheel in order to raiſe a weight y which is applied to the circumference of the axle, it is required to aſſign the quantity y, when the moment generated in it in any given time ſhall be the greateſt poſſible, the inertia of the wheel and axle not being conſidered.

Let $SD = d$, $SA = a$, the force which accelerates the aſcent * of $y = \frac{adp - ya^2}{d^2p + ya^2}$; therefore if $l = 193$ inches, the velocity generated in y in the † time t will be $2tl \times \frac{adp - ya^2}{d^2p + ya^2}$, and the moment generated in y in the ſame time t will be expreſſed by $2lt \times \frac{adpy - a^2y^2}{d^2p + ya^2}$; and as this is to be the greateſt poſſible by the problem, its fluxion muſt be $= 0$, which will give $ad^3p^2\dot{y} - 2a^2d^2py\dot{y} - a^4y^2\dot{y} = 0$, and the weight ſought $= \frac{\sqrt{d^4p^2 + d^3p^2a} - d^2p}{a^2}$.

Cor. 1. If SD is to SA as $n : 1$, $y = p \times \sqrt{h^4 + n^3 - n^2p}$: if the radius of the axle $=$ the radius of the wheel, that is, if $n = 1$, the weight y will be $= p\sqrt{2} - 1$: the weight moved therefore muſt be about $\frac{5}{12}$ parts of the moving force.

Cor. 2. If the wheel's weight be taken into account, let its weight $= w$, the diſtance of the centre of gyration from the axis $= r$, and the moment generated in the aſcending weight the time t will be expreſſed by $2lt \times \frac{adpy - a^2y^2}{wr^2 + d^2p + ya^2}$, which is the greateſt poſſible when $y = \frac{\sqrt{d^4p^2 + 2d^2pr^2w + r^4w^2 + pwdar^2 + p^2d^3a} - d^2p - r^2w}{a^2}$.

* Sect. VI. Prop. XIX.

† Sect. III. Prop. II.

XXII.

Fig. LXIII. Let A B C H be a ſyſtem of bodies moveable round a vertical axis which paſſes through the common centre of gravity of the ſyſtem. Suppoſe D E G to be a wheel the axis of which is vertical and coincident with that of the ſyſtem, let motion be communicated by means of a line going round this wheel, the ſtring D P being ſtretched by a given weight P; let it be required to aſſign the radius of the wheel E G D, ſo that the angular velocity communicated to the ſyſtem in any given time may be the greateſt poſſible.

Let the weight of the ſyſtem $= w$, the diſtance of the centre of gyration from the axis of motion $= r$, the radius ſought $SD = x$: then will the velocity generated in a
* Sect. VI. Prop. XV. given time in the * deſcending weight p be proportional to $\frac{px^2}{wr^2 + px^2}$, and the angular velocity generated in the ſame time as $\frac{px}{wr^2+px^2}$, which is to be a maximum by the conditions of the problem; we have therefore its fluxion $\frac{pwr^2\dot{x} + p^2x^2\dot{x} - 2p^2x^2\dot{x}}{\overline{wr^2 + px^2}^2} = 0$, and the diſtance ſought

$$x = \sqrt{\frac{w}{p}} \times r.$$

Thus, ſuppoſe the moving force is $\frac{1}{4}$ of the weight of the ſyſtem, it ſhould be applied at a diſtance from the axis equal to twice the diſtance of the centre of gyration, in order to produce the greateſt angular velocity in a given time.

In order to increaſe the action of a given moving force againſt a weight to be raiſed, or reſiſtance to be overcome, a combination of two or more mechanic powers is frequently made uſe of. Let *p* be a power applied by means of a line to the vertical wheel *C*: ſuppoſe the circumference of the axle *K* to be in contact with the circumference of any other vertical wheel *B*, ſo that the circumference of the wheel *B* may always move equally faſt with that of the axle which belongs to *C*: let alſo the axle of *B* communicate motion to a vertical wheel *A*, to the axle of which a weight *q* is ſuſpended, ſo as to act in oppoſition to *p*: moreover let *l m n* : 1 be the ſum of the ratios of the radius of each wheel to that of its axle; then if *p l m n* = *q*, the two weights *p* and *w* will ſuſtain each other in equilibrio, but if *l m n p* is at all greater than *q*, the equilibrium will be deſtroyed, and the next problem is intended to aſſign the motion communicated to *p* or *q* in given circumſtances. Fig. LXIV.

XXIII.

In a ſyſtem of wheels and axles juſt deſcribed, let the radii of the wheel and of the axle A be in the ratio of l : 1, the radii of the wheel and axle in B as m : 1, and of C as n : 1; alſo let the diſtance of the centre of gyration from the axis in A be r, the diſtance of the centre of gyration in B = s, in C = t: and let the weight of the wheel and axle A = a, that of B = b, and of C = c: ſuppoſe the weight p ſufficient to communicate motion to q; and let it be required to aſſign Fig. LXIV.

the

the ſpace deſcribed by q in its aſcent from reſt during a given time.

The abſolute force of p to move itſelf $= p$, and ſince q acts in oppoſition to this force, its effects muſt be ſubducted from p in order to obtain the force which impels p on the whole; and ſince q would balance a weight $= \frac{q}{lmn}$ if applied at p, the force which impels p on the whole will be $\frac{plmn - q}{lmn}$. In the next place, the inertia which reſiſts the communication of motion to p muſt be aſcertained. Motion is communicated to the wheel A from the circumference of the axle B, and the inertia of A and of the weight q which reſiſts the [*] communication of a force applied at $I = \frac{ar^2 + q}{l^2}$: in regard therefore to the inertia of A and q, theſe may be ſuppoſed to be removed, and the equivalent [†] maſs $\frac{ar^2 + q}{l^2}$ collected into the circumference of the wheel A, or of the axle B. Since motion is communicated to B by the circumference of the axle C, the inertia of B together with the equivalent maſs $\frac{vr^2 + q}{l^2}$ will be $\frac{s^2 l^2 b + ar^2 + q}{l^2 m^2}$: in like manner ſince motion is communicated to C by the weight p acting at D, the inertia which reſiſts the communication of motion to D or p will be $\frac{pn^2 + ct^2}{n^2} + \frac{s^2 l^2 b + ar^2 + q}{l^2 m^2 n^2} = \frac{l^2 m^2 n^2 p + ct^2 l^2 m^2 + s^2 l^2 b + r^2 a + q}{l^2 m^2 n^2}$, and the force which accelerates p in its deſcent from reſt $= \frac{\overline{plmn - q} \times lmn}{l^2 m^2 n^2 p + ct^2 l^2 m^2 + s^2 l^2 b + r^2 a + q}$, and that which accelerates $q = \frac{\overline{plmn - q}}{l^2 m^2 n^2 p + ct^2 l^2 m^2 + s^2 l^2 b + r^2 a + q}$. If therefore this force be put $= F$, and l be $= 193$ inches, the ſpace deſcribed by q in its aſcent from reſt in the time t ſeconds [‡] will be $= t^2 l F$ inches.

[*] Sect. VI. Prop. VIII.

[†] Sect. VI. Prop. I.

[‡] Sect. III. Prop. IV. Cor. 5.

XXIV.

Let A, B repreſent a ſingle moveable and a fixed pully, by means of which the power P elevates the weight W: having given P and W, together with the weights of the cylindrical pullies A and B, it is required to aſſign the ſpace which the deſcending weight P deſcribes in a given time, the weight of the moveable pully being included in the weight W.

Fig. LXV.

The abſolute force which impels P in its deſcent, is its own weight or P; but ſince the weight W acts againſt this deſcent with a force $= \frac{W}{2}$, the whole force by which P endeavours to deſcend is $P - \frac{W}{2}$. The inertia which reſiſts the communication of motion to P, depends on the maſs contained in P, together with that contained in the two pullies and the weight W. Let the weight of each pully be Q, the inertia of P is its own maſs P, and that of the pully‡ $A = \frac{Q}{2}$: the inertia of the pully B, is the ſame as if $\frac{Q}{2}$ were uniformly accumulated into its circumference; but ſince the velocity of this circumference is leſs than the velocity of P in the proportion of 2 : 1, its inertia will be equivalent to a ‖maſs $= \frac{Q}{2 \times 4}$ deſcending with P; for the ſame reaſon, ſince the weight W moves with a velocity only one half of that with which P deſcends, its inertia referred to P's motion will be $\frac{W}{4}$: the inertia of the whole maſs will therefore be $P + \frac{Q}{2} + \frac{Q}{8} +$

‡ Page 225.

‖ Sect. VI. Prop. II.

$\frac{W}{4} = \frac{8P+5Q+2W}{8}$; and becauſe the impelling force is $\frac{2P-W}{2}$, the accelerating force will $= \frac{2P-W}{2} \times \frac{8}{8P+5Q+2W} = \frac{8P-4W}{8P+5Q+2W}$: if therefore l be $= 193$ inches, the ſpace deſcribed by P from reſt in t ſeconds will be $\frac{8P-4W}{8P+5Q+2W} \times t^2 l$.

* Sect. VI. Prop. IV.

Cor. If W ſhould be greater than $2P$, W will deſcend, and by the ſame method of reaſoning, the force which accelerates the deſcent of W will be found $= \frac{2W-4P}{2W+5Q+8P}$.

XXV.

In a ſyſtem of pullies in which the ſame ſtring goes round all the pullies contained in two blocks: having given the power P and the weight W raiſed by it, together with the number of the pullies and the weight and figure of each, it is required to aſſign the force which accelerates the deſcent of P, the weight of the lower block being included in the weight W.

Let the number of pullies be n, and ſuppoſe them as before to be cylindrical, and that the weight of each $= Q$; then the abſolute moving force which acts at $P =$ its own weight P: but becauſe W acts at P in a contrary direction with a force $= \frac{W}{n}$, the force which upon the whole impels P in its deſcent $= \frac{Pn-W}{n}$.

Moreover, the inertia of P is its own maſs P, that of the weight W, the velocity of which $= \frac{1}{n}$ part of P's velocity ‡ will be $= \frac{W}{n^2}$, and becauſe the circumference of ‡ Sect. VI. Prop. II. the ſloweſt pully moves with the ſame velocity as the weight W, and its inertia is the ſame as if $\frac{Q}{2}$ were collected into its circumference, the inertia by which it reſiſts the deſcent of P will be $\frac{Q}{2n^2}$, the circumference of of the next pully revolving twice as faſt as the former, its inertia will be four † times as great, and will therefore be † Sect. VI. Prop. II. $= \frac{4Q}{2 \times n^2}$, and ſo on; the ſum of the inertia of the pullies being $= \frac{Q}{2n^2} \times \overline{1 + 4 + 9}$, &c. continued to n terms, which is $= \frac{Q}{2n^2} \times \frac{\overline{2n^3 + 3n^2 + n}}{6}$: the whole inertia therefore which reſiſts the communication of motion to P will $= P + \frac{W}{n^2} + Q \times \frac{2n^3 + 3n^2 + n}{12n^2} = \frac{\overline{12n^2P} + \overline{12W} + Q \times \overline{2n^3 + 3n^2 + n}}{12n}$, and the force which accelerates the deſcent of $P = \frac{12Pn^2 - 12Wn}{12n^2P + 12W + Q \times \overline{2n^3 + 3n^2 + n}}$.

XXVI.

Let ABCD repreſent a ſyſtem of pullies in which the ſtring that goes round each pully is fixed to the weight, as repreſented in fig. LXVI: having given the weight W, and the power which raiſes it P, together with the number and weights of the cylindrical and equal pullies A, B, C, D, &c. it is required to aſſign the force Fig. LXVI.

which accelerates the deſcent of the power P, the weights of the pullies B, C, D being included in P.

Let the weight of each pully be $\mathcal{Q}$, and ſuppoſe their number $= n$; then the abſolute force which urges P downwards is its own weight P: but ſince W acts againſt it with a force $= \frac{W}{2^n - 1}$, the force by which P endeavours to deſcend upon the whole will be $\frac{P \times \overline{2^n - 1} - W}{2^n - 1}$. The inertia of W which retards the motion of $P = \frac{W}{\overline{2^n - 1}^{\,2}}$, becauſe the velocity with which W aſcends is $\frac{1}{2^n - 1}$ part of the cotemporary * velocity of P: and ſince the velocity of the circumference of the pully A is equal to that with which W aſcends, the inertia of the pully A inferred to P's motion will be $\frac{\mathcal{Q}}{2 \times \overline{2^n - 1}^{\,2}}$, and becauſe the angular velocities of the pullies A, B, C, D are as 1, 3, 7, 15, &c. the inertia of B will be $= \frac{9\mathcal{Q}}{2 \times \overline{2^n - 1}^{\,2}}$, that of $C = \frac{49\mathcal{Q}}{2 \times \overline{2^n - 1}^{\,2}}$, and the inertia of all the pullies A, B, C and D, &c. which ariſes from their rotation, $= \frac{\mathcal{Q}}{2 \times \overline{2^n - 1}^{\,2}} \times \overline{1^2 + 3^2 + 7^2 + 15^2}$, &c. continued to n terms $= \frac{\mathcal{Q}}{2 \times \overline{2^n - 1}^{\,2}} \times \frac{2^{2n+2} - 3 \times 2^{n+2} + 3n + 8}{3}$. Moreover, the inertia which ariſes from the motion of the centres of gravity of the pullies B, C, D, &c. as they deſcend, will be found by the ſame reaſoning to be $\frac{\mathcal{Q}}{\overline{2^n - 1}^{\,2}} \times \frac{2^{2n} - 3 \times 2^{n+1} + 3n + 5}{3}$, which being added to the

* Sect. VI. Prop. II.

inertia before found, will give the inertia of the pullies $\frac{Q}{6\times\overline{2^n-1}^2}\times\overline{2^{2n+2}+2^{2n+1}-6\times2^{n+2}+9n+18}$: the whole inertia therefore which reſiſts the communication of motion to P during its deſcent $=P+\frac{W}{\overline{2^n-1}^2}+$

$$\frac{Q}{6\times\overline{2^n-1}^2}\times\overline{2^{2n+2}+2^{2n+1}-6\times2^{n+2}+9n+18}=$$

$$\frac{6P\times\overline{2^n-1}^2+6W+Q\times\overline{2^{2n+2}+2^{2n+1}-6\times2^{n+2}+9n+18}}{6\times\overline{2^n-1}^2},$$

and becauſe the moving force which acts on P during its deſcent is $\frac{P\times\overline{2^n-1}-W}{2^n-1}$, the force which accelerates $P=$

$$\frac{\overline{6P\times\overline{2^n-1}}-6W\times\overline{2^n-1}}{6P\times\overline{2^n-1}^2+6W+Q\times\overline{2^{2n+2}+2^{2n+1}-6\times2^{n+2}+9n+18}},$$

and the force which accelerates W in its aſcent $=$

$$\frac{6P\times\overline{2^n-1}-6W}{6P\times\overline{2^n-1}^2+6W+Q\times\overline{2^{2n+2}+2^{2n+1}-6\times2^{n+2}+9n+18}}.$$

In order to render the ſolutions of theſe laſt propoſitions leſs complicated, the weights of the moveable pullies have been included either in the weight W or the power P. In the ſyſtem deſcribed in the * propoſition juſt demonſtrated, there is this particular advantage, i. e. the weights of the pullies conſtitute a part of the moving force and conſequently facilitate the elevation of any weight to be raiſed by it, whereas the weights of the pullies in the other conſtructions act againſt the moving force. The weights of the pullies in the ſyſtem deſcribed in the propoſition have been ſuppoſed to be included in the power P, but as they are not equal in their effect on the raiſed weight, it may be neceſſary to mention in what manner their forces when referred to the power P are eſtimated. It is manifeſt, that the pully A being fixed contributes nothing to the moving power: alſo if the weight of each pully be Q, the pully B acts on W with a force $=Q$, the pully C with a force $=3Q$, and D with a force $=7Q$; if therefore the number of pullies be n, the moving force exerted by them on the weight to be raiſed $=Q\times\overline{1+3+7+15,\ \&c.}$ continued to $\overline{n-1}$ terms $=Q\times\overline{2^n-1-n}$, and this

* Prop. XXVI.

this weight would be balanced by $\frac{Q\times\overline{2^n-1-n}}{2^n-1}$ acting at P, wherefore if a power G be applied at P, excluſive of the weights of the pullies, the whole moving force which acts on P will be $G+\frac{Q\times\overline{2^n-1-n}}{2^n-1}-\frac{W}{2^n-1}$.

If this value for the moving force be ſubſtituted for P in the preceding ſolution, that is, as far as that body is concerned in contributing to the moving force, we ſhall have a general expreſſion for the acceleration of the deſcending weight, every circumſtance being taken into account, except thoſe of friction, the line's weight, and the air's reſiſtance; the two latter of which are too ſmall to have ſenſible effect, and the former may be ſo diminiſhed, that the real motion of the deſcending weight ſhall ſcarcely deviate from that which is deduced from the theory. Let therefore G be any moving force applied to raiſe the weight W by means of the ſyſtem of pullies deſcribed in the propoſition, every thing elſe remaining, we ſhall have the force which accelerates the deſcending weight G equal to that part of gravity expreſſed by the fraction

$$\frac{6G\times\overline{2^n-1}+6Q\times\overline{2^n-1-n}-6W\times\overline{2^n-1}}{6G\times\overline{2^n-1}^2+6W+Q\times\overline{2^{2n+2}+2^{2n+1}-6\times2^{n+2}+9n+18}}.$$

In the ſame manner the motion generated in any of the mechanic powers may be aſcertained from the neceſſary data, and in other caſes, provided the forces by which each particle of the ſyſtem is moved be conſtant. Although the following propoſition be not immediately connected with the ſubject of rotatory motion, yet as the ſolution of it is included in a more general theory hereafter demonſtrated, of which the principles of rotation are likewiſe particular caſes, it may be not uſeleſs to inſert it.

XXVII.

Fig. LXVII. Let ABC be an iſoſceles wedge, the baſe of which is AB; let two equal ſpheres be applied ſo as to touch the ſides in the points D and E equally diſtant from C: then the baſe being biſected in G let a weight P be applied ſo as to urge the wedge forward in the direction G C H, and to communicate motion at the ſame time to the ſpheres D and E in free ſpace, the wedge and ſpheres being conſidered without gravity, and their ſurfaces being perfectly ſmooth: having given the neceſſary conditions, it is required to aſſign the force which accelerates the weight P and the ſpheres D and E, and the direction in which the ſpheres move.

Let the maſs contained in the wedge $= W$, that in either ſphere $= Q$, then will the force which tends to communicate motion to the weight P be $= P$; to find the maſs moved, ſince each particle of the wedge moves equally faſt with P, its inertia will be $= W$: now to find the inertia of the ſphere D, ſuppoſe the wedge to have moved through a ſpace $= CH$, draw HI parallel to AC, and HK to CB, then if DI be drawn perpendicular to HI, and EK to HK, when the point C arrives at H, the ſpheres D and E will be coincident with I and K reſpectively: and if IA, KB be drawn parallel to CG, the point A will become coincident with the ſphere D at I, and the point B with the ſphere E at K: it follows, that the force which impels the ſphere D will be equal to that part of the force P which is expreſſed by the fraction $\frac{AG}{AC}$, and the velocity of D will be that part of P's cotemporary velocity which is expreſſed by the fraction $\frac{DI}{IA}$ or $\frac{AG}{AC}$. The inertia therefore of the ſphere D whereby it reſiſts the communication of motion* to P is $= \frac{Q \times AG^2}{CA^2}$, and the inertia of both the ſpheres $= \frac{2Q \times AG^2}{CA^2}$: wherefore ſince the force of P to move itſelf $= P$, and the inertia exerted againſt the communication of motion to it $= P + \frac{2Q \times AG^2}{CA^2} + W$, the force which accelerates P will be $= \frac{P \times CA^2}{P \times CA^2 + W \times CA^2 + 2Q \times AG^2}$‡, and that which accelerates the ſpheres D, E will $= \frac{P \times AG \times CA}{P \times CA^2 + W \times CA^2 + 2Q \times AG^2}$.

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* Sect. VI. Prop. II.

Fig. LXVIII. Let a body be moved from reſt at A in the direction AB by any conſtant force; then if the force ceaſes to act at B, the body will proceed with the ſame uniform velocity which it had acquired at B. Suppoſe when it arrives at C ſome obſtacle or reſiſtance is oppoſed to its progreſs: if the oppoſition ariſes from an inert and nonelaſtic obſtacle on which it impinges, it will after a || certain time, ſtill proceed uniformly through with a diminiſhed velocity. If inſtead of an inert obſtacle oppoſed to the body's progreſs at C, it be acted upon by an uniformly reſiſting force, the body will be continually retarded until its motion is deſtroyed. Moſt mechanical operations conſiſt either in communicating motion to quieſcent and inert bodies or in overcoming reſiſtances, in which operations the action of the moving force and its mechanical effect are not always cotemporary; but motion is frequently firſt generated, and at ſubſequent times employed in producing the deſired reſult. Thus, in the preceding example, motion is generated while the body is deſcribing AB, and it is ſubſequently employed in overcoming the reſiſtance oppoſed while the body deſcribes ſome ſpace CD. The mechanical effect of the motion generated will be greater or leſs according to various circumſtances: for a greater maſs of matter will overcome a greater reſiſtance than a ſmaller moving with the ſame velocity, every thing elſe being equal: it is evident that the effect will depend on the velocities alſo, but the exact quantity of motion communicated to a body by a moving force or deſtroyed by any reſiſtance, cannot be eſtimated by the quantity of matter contained in the body and its velocity, neither by aſſuming the quantity of motion as the maſs into the ſquare of the velocity, nor as the quantity of matter into the velocity, and for theſe reaſons; there are five quantities concerned in the generation of motion, i. e. the quantity of matter moved, the

force

‡ Sect. I. Prop. IX.

|| Sect. I. p. 8. and Sect. III. p. 48.

force which moves it, the ſpace deſcribed, time of deſcription, and velocity generated, and any two of theſe being given will not determine the reſt or any one of them. If therefore we were to affirm, either that $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{Q}{q}$, or that $\frac{M}{m} = \frac{V}{v} \times \frac{Q}{q}$, either affirmation would be as inconcluſive, as if eſtimating the ſpaces deſcribed by bodies moving with uniform velocities, we were to ſay, that the ſpaces deſcribed are in the ſame ratio as the velocities of motion; but any three of the five quantities juſt mentioned being given determine the reſt: wherefore if $\frac{S}{s}$ be the ratio of the ſpaces deſcribed, and that of the times of motion be $\frac{T}{t}$,* ſince $\frac{M}{m} = \frac{Q}{q} \times \frac{V}{v} \times \frac{t}{T}$, if $T = t$, then $\frac{Q}{q} \times \frac{V}{v} = \frac{M}{m}$; that is, the ratio of the moving forces is equal to the ſum of the ratios of the velocities generated, and of the quantities of matter moved when the times are the ſame: moreover, ſince † $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{Q}{q} \times \frac{s}{S}$, if $\frac{V^2}{v^2}$ and $\frac{Q}{q}$ be given, when $S = s$, $\frac{M}{m}$ is determined, and conſequently when the ſpaces deſcribed are equal, the ratio of the moving forces $= \frac{V^2}{v^2} \times \frac{Q}{q}$. Now it is certain, that we are at liberty to eſtimate mechanical effects and the forces by the operation of which they are produced, by the velocities deſtroyed or generated in bodies, during their motion through a given ſpace or in a given time; and according to which of theſe hypotheſes is aſſumed, the moving forces or reſiſtances will be as the ſquares of the velocities into the quantities of matter moved, or as the velocities into the quantities of matter. Thus Leibnitz, Bernoulli, and many of the foreign philoſophers, eſtimated the motion of bodies according to the former ſuppoſition; Newton, Keil, and moſt of the Britiſh, according to the latter: the ſame concluſions will of courſe be produced by each method from given data; in regard to the truth of theſe concluſions therefore, it is a matter of indifference which method be uſed in the inveſtigation of them, al-

* Sect. III. Prop. VII.

† Sect. III. Prop. VI.

although particular problems may come out in a more eaſy and ſimple manner from one hypotheſis than the other.

Theſe conſiderations being premiſed, when any mechanical effect is to be produced, in order to proportion duly the means to the end required, it muſt be conſidered, that if only one mechanic quantity be given in the problem, there are four others remaining, two of which may be varied ſine limite, and combined with all the different conſtructions which mechanic inſtruments are capable of. Thus, ſuppoſe it were required only to put in motion a given body *A*: of the force, the ſpace deſcribed, the velocity communicated, and the time of motion, any two may be infinitely varied, but the reſt are determined when any two with the original quantity are given. 2. Suppoſe it were required to communicate to a given body *A* a certain velocity *V*, then of the force, the ſpace deſcribed by *A* before it has acquired the given velocity as well as the time of acceleration, any one of theſe may be aſſumed of any magnitude at pleaſure. In the ſame manner, if it were required to ſtop or deſtroy the motion of a body *A*, the velocity of which is *V*; here a great force may ſtop the body in a ſmall time, or a ſmall force in a great time, the ſpace being determined by either of theſe quantities; but the ſame effect, i. e. the deſtroying the motion of *A*, is obtained in any caſe, however the means may vary: if it be required to generate a velocity *V* in a given body *A* during the time *T*, the force to be applied as well as the ſpace deſcribed, before *A* has acquired that velocity ceaſe to be a matter of choice; there being but one force that can be applied to ſatisfy the conditions of the problem which limits the ſpace deſcribed alſo; and the ſame method of reaſoning is applicable to the deſtroying the motion of *A* by the action of an uniformly reſiſting force.

From obſerving theſe circumſtances, an uſeful problem in mechanics may be ſolved, ſuch as having given the reſiſtance overcome by a given body moving through a given ſpace, to aſſign the quantity of matter contained in ſome other body moving with a given velocity which ſhall overcome the ſame reſiſtance, or produce a mechanical effect equal to the former, under certain conditions and reſtrictions required by the nature of the caſe.

XXVIII.

Having given the quantity of matter contained in a body = Q, and its velocity = V; then ſuppoſe it to be reſiſted by a force which is to that which gravity would oppoſe to the ſame body, when thrown perpendicularly upwards, as M : Q, and let the ſpace deſcribed by Q before its motion is wholly deſtroyed by the reſiſting force M be S, the whole time of its motion being = T; let it be required to aſſign what muſt be the quantity of matter q moving with any other given velocity u, ſo that the reſiſtance m ſhall deſtroy its whole motion: 1. while it deſcribes the ſame ſpace as before S; 2. in the given time T.

Since univerſally* $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{Q}{q} \times \frac{s}{S}$, when $S = s$ and $M = m$, as in the firſt part of the problem, $\frac{V^2}{v^2} = \frac{q}{Q}$, wherefore the weight required $q = \frac{Q \times V^2}{v^2}$, when the ſame ſpace is deſcribed as in the former caſe. 2d. ‡ Since in general $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{Q}{q} \times \frac{t}{T}$, when $T = t$ and $M = m$, as in the ſecond part of the problem, it follows that $\frac{V}{v} = \frac{q}{Q}$, wherefore the weight ſought $q = \frac{V}{v} \times Q$.

* Sect. III. Prop. VI.

‡ Sect. III. Prop. VII.

Let an example be taken from Mr. Robins[†]: he found that a leaden ball of $\frac{3}{4}$ of an inch diameter, impinging perpendicularly with a velocity of 1700 feet in a ſecond, on an elm block penetrated into its ſubſtance 5 inches: let it be required to aſſign what quantity of matter muſt impinge on the ſame ſubſtance with a velocity of one foot in a ſecond, ſo that its whole motion ſhall be deſtroyed while it deſcribes the ſame ſpace, that is, 5 inches: here referring to the ſolution we have $V = 1700$, $\mathcal{Q} = 1$, $v = 1$, and the maſs required $q = \frac{V^2}{v^2} \times \mathcal{Q} = \overline{1700}^2 \mathcal{Q}$, and $q = 2890000$ times the ball's weight. This is the ſolution when the ball's motion is deſtroyed during the time it deſcribes 5 inches in the elm, being the whole depth to which it penetrates. If it be required to aſſign the maſs, when the ball's motion is deſtroyed in the ſame time as before, the quantity will be $\frac{V}{v} \mathcal{Q} = 1700$ times the ball's weight.

Fig. LXIX.

It remains to ſhew in what manner this theory may be practically illuſtrated. Let I repreſent the bullet, ſuch as is referred to in the preceding example, and which would if it impinged on a block of elm with a velocity of 1700 feet in a ſecond, penetrate into its ſubſtance to the depth of 5 inches. In the preceding problem, when the maſs contained in the ball is increaſed, in order to compenſate for the diminiſhed velocity, ſo that its effects ſhall be the ſame, the reſiſtance of the block m is ſuppoſed to be neither greater or leſs than before: it follows from theſe conſiderations, that in augmenting the quantity of matter contained in the ball, the magnitude of it muſt not be altered; for if it were, the [‡] reſiſtance of the elm would not continue the ſame. The following means may be made uſe of, whereby the magnitude of the ball will not be altered, and yet the impulſe exerted by it when impinging againſt an elm block with the given velocity of one foot in a ſecond ſhall be the ſame as if it were in fact 2890000 times heavier than before. Suppoſe the ball I affixed to an iron rod, which is faſtened into a heavy block of wood or other ſubſtance, the weight of which is 2890000 times greater than that of the ball I: and ſuppoſe this block ſo ſuſpended as to revolve about an horizontal axis of motion LM at a convenient height, and which is parallel to the plane of the elm block. Let the rod ID paſs through the centre of percuſſion, the pendulum vibrating in a plane perpendicular to the axis: then let the block be brought to

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[†] Robins' Gunnery, Vol. I.

[‡] Sect. III. p. 38.

ſuch a poſition, that it ſhall touch the bullet when the pendulum is quieſcent, *ID* being horizontal: alſo let the pendulum be drawn out of its vertical poſition by any force through an arc the verſed ſine of which is equal to $\frac{1}{64}$ part of a foot: if it be ſuffered to deſcend from reſt along this arc to its loweſt point, it will there have acquired a velocity of one foot in a ſecond, and if the block be immoveably fixed, the whole motion of the impinging body will be deſtroyed. Moreover, by the problem, the depth to which the bullet *I* will penetrate into the ſubſtance of the block will be 5 inches. The ſame mechanical effect is therefore produced by an inconſiderable, as by a very great velocity, compenſation being made for the diminiſhed velocity by increaſing the maſs of matter moved. And it is plain that the ſame means may be uſed to produce by vaſt and maſſy bodies moving with ſmall velocities, the effects of cannon balls impinging againſt obſtacles conſiſting of ſtone walls, earth, or wood. Theſe two methods have been made uſe of in different ages for the purpoſe of demoliſhing fortifications: the battering rams of the ancients conſiſted of very large beams of wood, terminated by ſolid bodies of iron or braſs; ſuch a maſs being ſuſpended as a pendulum, and driven partly by its gravity and partly by the impulſe of men againſt the walls of a fortification, exerted a force which in ſome reſpects exceeded the utmoſt effects of our battering cannon, though in others it was probably inferior to the modern ordnance. To compare the effects of a battering ram, the metal extremity of which ſuppoſe equal in magnitude to a 24 pounder, with that of a cannon ball of 24 pounds weight: in order that the two bodies may have the ſame effect in cutting a wall, or making a breach in it, the weight of the aries muſt exceed that of the ball in the proportion of about $\overline{1700}^{2}$[*] to the ſquare of the velocity with which the battering ram could be made to impinge againſt a wall expreſſed in feet; if this may be eſtimated at about 10 feet in a ſecond, the proportion of the weights will be that of about 2890000 to 100, or 28900 to 1: the weight of the battering ram muſt therefore = 346 ton. In this caſe the battering ram and the cannon ball moving with the velocities of 10 feet and 1700 feet reſpectively in a ſecond, would have the ſame effect in penetrating the ſubſtance of an oppoſed obſtacle; but it is probable, that the weight of the aries never amounted to ſo much as is above deſcribed, and conſequently the effects of the cannon ball to cut down walls by making a breach in them, muſt exceed thoſe of the

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[‡] Sect. VI. p. 209.

[*] Sect. VI. Prop. XXIV.

ancient battering rams: but the momentum of theſe, or the impetus whereby they communicated a ſhock to the whole building was far greater than the utmoſt force of cannon balls; for if the weight of the battering ram were no more than 170 times greater than that of a cannon ball, each moving with its reſpective velocity, the moments of both would be equal: but as it is certain, that the weight of theſe ancient machines was far more than 170 times our heavieſt cannon balls, it follows, that their moment or impetus to ſhake or overturn walls, &c. was far ſuperior to that which is exerted by the modern artillery. And ſince the ſtrength of fortifications will in general be proportioned to the means which can be uſed for their demolition, the military walls of the moderns have been conſtructed with leſs attention to their ſolidity and maſſy weight, than the ancients thought a neceſſary defence againſt the aries: that ſort of coheſive firmneſs of texture which reſiſts the penetration of bodies being now more neceſſary than in ancient times; but it is manifeſt, that even now, ſolidity or weight in fortifications alſo is of material conſequence to the effectual conſtruction of a wall or battery. This remark has been urged only to ſhew ſuch variations in the degrees of ſolidity and firmneſs of texture in the fabric of military walls, as have been occaſioned by the change which took place in the practice of artillery, when lighter bodies impelled with greater velocities were ſubſtituted inſtead of the ponderous machines of the ancients, to which but inconſiderable velocities could be communicated.

Fig. LXX. There is alſo another method of producing great mechanical effects by means of ſmall velocities generated in ponderous bodies by inconſiderable moving forces. Suppoſe *ABC* were an heavy cylinder of iron or lead, moveable about its axis and in a vertical plane: a ſmall force being applied to turn the cylinder, if long continued, will generate ſuch a force in it as will produce effects in raiſing weights and overcoming reſiſtances, by no means obtainable by the moving force immediately applied. Suppoſe, for inſtance, it were required to aſſign what muſt be the weight of the wheel, ſo that when the bullet referred to in the laſt example is affixed to any point *D*, and revolves with the cylinder at a given rate, for example, with a † velocity of 1 foot in a ſecond, if it ſhould impinge perpendicularly againſt an immoveable block of elm, it ſhall penetrate into that ſubſtance to the ſame depth as when it impinged freely upon the immoveable block with a velocity of 1700 feet in a ſecond.

† Page 265.

cond. Here the velocity of the ball = 1 foot in a ſecond, let p = its weight, and x the weight of the cylinder ſought; moreover, let R be the centre of gyration of the cylinder, then will the effect of D ſtriking againſt an obſtacle be the ſame as if the whole weight of the wheel were removed, and the equivalent *maſs $\frac{x \times SR^2}{SD^2}$ collected into the point D. And ſince p or D's velocity is that of one foot in a ſecond, the maſs at D muſt be $= p \times \overline{1700}^2$, or 2890000 || times the weight of p, putting therefore $p \times 2890000 = \frac{x \times SR^2}{SD^2}$, we have the weight of the cylinder, which will anſwer the conditions of the problem, $= 2890000p \times \frac{SD^2}{SR^2}$; and if the radius of the cylinder $= 10 \times SD$, then will $SR^2 = 50 \times SD^2$†, and the cylinder's weight $= \frac{2890000p}{50} = 57800p$, that is, if $p = 1.3048$ oz. the weight would become = 4713 pounds.

* Sect. VI. Prop. VIII.

|| Supra.

† Sect. VI. p. 224.

It remains therefore only to aſſign for what time or through what ſpace a moving force, which may be aſſumed of convenient magnitude, muſt act on the wheel, ſo that a velocity of 1 foot in a ſecond may be generated in the point D, the diſtance of which from the axis is $\frac{1}{10}$ part of thee wheel's radius. The ſolution of this may ſerve as an example to the principles already demonſtrated, as well as for the purpoſe of illuſtrating the ſubject in queſtion: the moving force may be aſſumed equal to that which the human body can apply without any remarkable exertion.

XXIX.

Let a cylinder of any given radius, for example 10 feet, the weight of which = 4713 pounds, revolve round an horizontal axis of motion; it is required to aſſign how long a moving force of 20 pounds

Fig. LXX.

weight muſt act on the circumference H, in order to generate a velocity of one foot in a ſecond in the point D, SD being = 1 foot.

When $SD = 1$ foot let SR be the diſtance of the centre of gyration from the axis, and when the velocity in D is that of one foot in a ſecond, the velocity of the circumference will be 10 feet in a ſecond; and ſince the cylinder's weight will reſiſt the communication of motion to the circumference in the ſame manner, as if the whole maſs were removed and the equivalent maſs $\frac{4713 \times SR^2}{SD^2}$ were collected into the circumference, that is,* if $\frac{4713}{2} =$ 2356.5 pounds were ſo collected, we ſhall have the force which accelerates the circumference $= \frac{20}{20 + 2356.5} = \frac{1}{118.8}$, and the time in which this force will generate a velocity of 10 feet in a ſecond $= \frac{10 \times 118.8}{32\frac{1}{6}} = 37$ ſeconds. A weight therefore of 20 pounds acting for 37 ſeconds at the circumference of ſuch a cylinder as is deſcribed in the problem, will generate a moment which being accumulated in a muſket ball fixed at the diſtance of $\frac{1}{10}$ the cylinder's radius from the axis will produce an effect in penetrating an oppoſed block of elm immoveably fixed, equal to that which is exerted by the ſame muſket ball fired with its full charge of gunpowder againſt the ſame block.

It muſt be here obſerved, that the ſtrength of the human body cannot exert conſtantly ſo great a quantity of force as that of 20 pounds, if it be applied ſo far from the axis as 10 feet: the velocity of 10 feet in a ſecond is much greater than that which the velocity of the arm could keep up to, for when the velocity generated is no more than 4 or 5 feet in a ſecond, a perſon by pulling can ſcarcely keep the rope ſtretched with a force equal to 20 pounds: but then

* Sect. VI. p. 224.

it may be urged, that the force which a man can exert by pulling downward, when the velocity of the maſs moved is inconſiderable, is nearly equal to his weight, and therefore may in ſome meaſure compenſate for the deficiency juſt deſcribed; but at all events the ſolution may be accommodated to any particular caſe. For inſtance, let the wheel be turned by a handle, the diſtance of which from the axis = 1 foot, then will the force which accelerates the circumference be * diminiſhed in the ratio of 10 : 1, and the time of generating the given angular velocity above deſcribed will be † increaſed in the ratio of 10 : 1, and ſo will become 370 ſeconds, or 6 minutes 10 ſeconds, provided a force equal to 20 pounds be conſtantly exerted.

* Sect. VI, Prop. III.

† Prop. XVI. Cor. 5.

This accumulation of mechanic force appears extraordinary at firſt ſight, and might ſuggeſt wrong notions concerning the ſubject unleſs fully conſidered. It might ſeem from this great power gained, that by increaſing the time in which the moving force acts, the moment communicated to a ponderous wheel might be applied to elevate a weight to an altitude greater than that from which an equal weight deſcended from reſt, in order to generate the motion in the ſyſtem; and were this really the caſe, nothing more would be wanting to conſtitute a machine containing the principle of perpetual motion within itſelf: but it will appear from the enſuing propoſitions, that the altitude to which a weight is elevated by a power which communicates motion to a revolving ſyſtem, and is any how applied to raiſe it, can never be ſo great as that from which an equal weight muſt deſcend, in order to generate the motion ſo applied: the times wherein the weights deſcend and aſcend not being of any conſequence, the propoſition depending on the altitudes only.

XXX.

Fig. LXXI.

Let ABC repreſent a ponderous cylinder moveable in a vertical plane about its axis: let motion be communicated to the whole by a weight p deſcending through a ſpace s; when the action of p ceaſes after it has deſcended through the ſpace s, the ſyſtem will continue to revolve uniformly; let the moment thus generated be employed to elevate another weight q: the propoſition affirms, that the ſpace through which the wheel's motion elevates q, can never be ſo great as $s \times \frac{p}{q}$, but will be ultimately equal to $s \times \frac{p}{q}$ when the maſs of the revolving body is increaſed ſine limite, the effects of friction not being here conſidered.

There are two caſes contained in this propoſition; firſt, the weight q when quieſcent is acted on by the revolving ſyſtem: 2dly, the weight q always poſſeſſes motion equal to that of the circumference to which it is applied. To conſider both theſe; let the radius $SD = d$, $SE = a$, the centre of gyration $= R$, $SR = r$, the weight of the revolving body $= w$, the ſpace through which p deſcends from reſt $= s$. 1ſt. Let p by deſcribing the ſpace s generate motion in the ſyſtem before q is applied at the circumference EH: then ſince the force which accelerates the deſcent of $p = \frac{pd^2}{pd^2 + wr^2}$, the velocity acquired by p during its deſcent through the ſpace * $s = \sqrt{\frac{4lspd^2}{pd^2 + wr^2}}$, l being $=$ 193 inches, and the velocity generated in E at the ſame || inſtant $= \sqrt{\frac{4lspa^2}{pd^2 + wr^2}}$. Now, ſuppoſe p to be removed, at that inſtant let E be connected with q while quieſcent, then will the wheel have the ſame effect in communicating motion to q as if the whole maſs were removed, and the ‡ equivalent maſs $\frac{w \times r^2}{a^2}$ were collected into

* Sect. VI. Prop. XV. Cor. 5.

|| Sect. VI. Prop. III.

‡ Sect. VI. Prop. VIII.

into *E*. In order therefore to determine the motion of q, the laws of colliſion give us this proportion $\frac{wr^2}{a^2} + q :: \frac{wr^2}{a^2} :: \sqrt{\frac{4lspa^2}{pd^2 + wr^2}}$ to the common velocity with which *E* and q begin to move, which will therefore $=$ $\sqrt{\frac{4lspa^2}{pd^2 + wr^2}} \times \frac{wr^2}{wr^2 + a^2q}$: and ſince the conſtant force which always retards the aſcent of $q = \frac{qa^2}{wr^2 + qa^2}$, the height S to which q will † aſcend will $=$ $\frac{4lspa^2}{4l \times \overline{pd^2 + wr^2}} \times \frac{w^2r^4}{\overline{wr^2 + qa^2}^2} \times \frac{wr^2 + qa^2}{qa^2} = \frac{sp}{q} \times \frac{w^2r^4}{\overline{pd^2 + wr^2} \times \overline{wr^2 + qa^2}}$, which is always leſs than $\frac{sp}{q}$: but approaches $\frac{sp}{q}$ as a limit when wr^2 is increaſed or $pd^2 + qa^2$ diminiſhed ſine limite.

† Sect. III. Prop. V. Cor. 3.

In the preceding caſe, motion has been ſuppoſed to be generated in the ſyſtem *ABC* by a weight p deſcending through a given ſpace s; and this motion being communicated to a quieſcent weight q, applied at any diſtance whatever from the axis, has been ſhewn inſufficient to elevate q to an altitude which is to s, the ſpace deſcended through by the weight p, in ſo great a proportion as that of $p : q$. There is ſtill another caſe to be conſidered: ſuppoſe the weight q to begin its motion with the revolving ſyſtem, being elevated by the deſcent of $\overline{q + p}$, p being here alſo the moving force: then let $\overline{q + p}$ deſcend through a given ſpace s; it is to be proved that if $\overline{q + p}$ be removed the inſtant the ſpace s has been deſcribed, the velocity generated in the ſyſtem will not be ſufficient to elevate q to an altitude ſo great as $s \times \frac{p}{q}$. Let the weight q acquire gradually from quieſcence a velocity equal to that of the point *E*: the force which accelerates p in its deſcent $= \frac{pd^2}{qa^2 + \overline{q + p} \times d^2 + wr^2}$, and the velocity generated in p during its deſcent through a ſpace $s = \sqrt{\frac{4lspd^2}{wr^2 + qa^2 + \overline{q+p} \times d^2}}$, and the velocity generated in $E = \sqrt{\frac{4lspa^2}{wr^2 + qa^2 + \overline{q+p} \times d^2}}$. Now let $q + p$ be removed; then will the weight q begin to aſcend with the velocity $\sqrt{\frac{4lspa^2}{wr^2 + qa^2 + \overline{q+p} \times d^2}}$, and being retarded by the conſtant force $\frac{qa^2}{wr^2 + qa^2}$, the ſpace S through which q will riſe $= \frac{spa^2}{wr^2 + qa^2 + \overline{q+p} \times d^2}$ $\times \frac{wr^2 + qa^2}{qa^2} = \frac{sp}{q} \times \frac{wr^2 + qa^2}{wr^2 + qa^2 + \overline{p+q} \times d^2}$ which is always leſs than $s \times \frac{p}{q}$ while the quantities w, r, &c. are finite: but approaches $s \times \frac{p}{q}$ as a limit while wr^2 is increaſed, or $\overline{p+q} \times d^2$ is diminiſhed ſine limite.

It appears from both caſes, that when any motion is generated by the deſcent of a heavy body, and this motion is afterwards employed to elevate another weight, the altitude of the elevated weight into its quantity of matter, will be always leſs than the altitude from which the former body deſcended multiplied into the body itſelf. If therefore this operation were repeated, and the motion laſt generated were applied to elevate another weight v by the deſcent of q, the motion generated by the prior deſcent would be diminiſhed; this diminution being the conſequence of every ſucceſſive elevation, until the motion is entirely deſtroyed.

This ſeems to ſtrengthen thoſe arguments which have been urged to ſhew the impoſſibility of a machine poſſeſſing the [illegible] principle of perpetual motion within itſelf: and it is worthy of remark, that the continual loſs of motion above demonſtrated, is wholly independent of the effects of friction, and ariſes from the inertia of the ſyſtem only; and this appears to be in fact the greateſt obſtacle which prevents the conſtruction of a ſelf mover: for, notwithſtanding what has been urged concerning the effects of friction and the air's reſiſtance, in order to account for the failure of the very many attempts to execute this famous project, it does not ſeem that they are ſuch very material impediments to the ſucceſs of it: for if it were poſſible from the principles of mechanics, independent of all obſtacles, to generate motion in a body by the deſcent of a weight through any ſpace, ſo that this motion when applied to raiſe a weight equal to the former, could elevate it to an altitude in the ſmalleſt degree greater than that from which the former weight deſcended; the reaſon does not appear why the ſame means by which this ſmall mechanical advantage was gained, ſhould not be employed in accumulating a ſtill greater acceſſion of mechanic force, ſuch as would be more than ſufficient to overcome the effects of friction and other reſiſtances.

The motion generated in ponderous wheels is applied to many practical purpoſes. Thus, machines to lift weights, to grind corn, &c. as well as many others, having a large wheel of maſſy ſubſtance affixed to ſome part of them, ſo that it ſhall revolve round its axis, have been found to produce far greater mechanical effects than when they are without ſuch additional maſs of matter.

Fig. LXXII.

Let ABC repreſent a wheel and axle moveable round an horizontal axis, paſſing through S by means of an arm or handle E, which is impelled by muſcular force: let w be a weight raiſed by the machine; and let $SD = d$, $SF = a$: part of the moving force applied at E or F to turn the wheel will be employed to balance w. Suppoſe, for example, the whole force applied at E were p, then will that part of it which is expreſſed by $\frac{Wa}{d}$ be employed in balancing w, and the other part $\frac{\overline{pd - aW}}{d}$ will communicate motion to the weight w: this is upon a ſuppoſition, that the force impreſſed on the handle E is conſtant. But this is not always the caſe: for in the firſt place, ſince the force which acts at E is the preſſure exerted by muſcular force, it is manifeſt, that when the velocity generated in E is equal to that with which the muſcular power could move if no ways impeded, all preſſure muſt ceaſe: moreover, in the intermediate velocities a given exertion of muſcular force will always preſs with leſs effect according as the velocity of the handle is greater; for which reaſon there is ſome velocity of the handle which will be continued uniform: becauſe if it be greater the moving force diminiſhes, and the force of the weight w preponderates againſt it; if it be leſs than that mean velocity, the moving force will accelerate it until it comes at the degree of motion above deſcribed. This being the caſe, the uſe of a ponderous wheel *ABC*, affixed ſo as to revolve on its axis with the ſyſtem is manifeſt: for, ſuppoſe a certain uniform velocity generated in it, this will continue for ſome time to elevate the weight w, although the moving force of the arm be diſcontinued, which muſt in ſome degree happen when the arm is aſcending; but if there were no motion in the wheel *ABC* to continue the aſcent of w, the weight w would begin to preponderate as ſoon as the moving force was at all diminiſhed, from which it is manifeſt how much motion muſt be continually loſt. And theſe arguments are further ſtrengthened by remarking, that a given muſcular force will produce by gradual acceleration as great an uniform velocity when the heavy wheel is applied to the machine, as when the machine is without it, the effects of friction not being conſidered; the only difference is in the time wherein this velocity is produced, the time being increaſed by the wheel's weight.

In the preceding propoſitions the force by which bodies or ſyſtems of bodies are cauſed to revolve, has been ſuppoſed conſtant; and the properties relating to gyration which have been demonſtrated are true, when the forces are variable, provided the time of their action be taken evaneſcent: from which concluſion, and having given the law according to which the force varies, the effects produced by a variable force on a revolving ſyſtem may be inveſtigated. In the laſt example, where rotatory motion is ſuppoſed to be generated in a ponderous wheel by the action of muſcular ſtrength, (of the human arm for inſtance,) it was obſerved, that the moving force was not conſtant, being greateſt at the firſt inſtant of motion, and abſolutely evaneſcent when the velocity of that part of the wheel, to which the hand was applied, became equal to the greateſt velocity with which the hand could move when not impeded.

If the laws according to which muſcular force varies in reſpect of the velocity of the body to which it is applied were known, the effects produced might be computed; but as this kind of power is irregular in its action, being in general greater as the point to which it is applied is deſcending, geometrical reaſoning cannot be uſed to eſtimate the motion generated by it, although there would be perhaps little difficulty in forming rules which ſhould be ſufficiently near the truth. In the next problem, rotation is generated by the action of a moving force, upon a principle not entirely diſſimilar to that juſt deſcribed; but the laws according to which the force varies in this caſe being known, (ſetting aſide the effects of friction) a ſolution may be obtained and will be a proper illuſtration of this part of the ſubject.

XXXI.

Let A B C repreſent a water wheel which revolves round an horizontal fixed axis, paſſing through its centre S. Suppoſe D E F to be the axle of this wheel, and that a weight W is affixed to a line D W ſo wound round the axle, that while the wheel is driven round in its own plane by the force of the water impinging at I, the weight W may be raiſed in a vertical line: having given the area of the boards I, I, I, againſt which the ſtream impinges perpendicularly, and the altitude from which the water deſcends, it is required to aſſign the greateſt velocity with which the wheel can revolve. Fig. LXXIII.

When a ſtream of any fluid impinges perpendicularly againſt a plane and quieſcent ſurface, the exact quantity of the moving force is equal to the weight of a column of the fluid, the baſe of which is the area upon which the fluid impinges, and the altitude that from which a body muſt deſcend freely from reſt by gravity to *acquire that velocity: this will be the moving force which impels the body when quieſcent, or juſt beginning to move, but after ſome motion has been communicated to the body upon which the fluid impinges, the impulſive force of the fluid will be diminiſhed; being the ſame as if the body were quieſcent, and the water impinged upon it with the ||difference of the former velocities. Wherefore the altitude of the column of the fluid, which is equal to its impelling force

* Sect. V. Prop. II.

|| III. Law of Motion.

force, will be always as the difference between the velocity of the impact and that of the body itſelf; and ſince the altitudes from † which bodies fall from reſt are in a duplicate ratio of the velocities acquired, it follows, that the force of the impact will be in a duplicate ratio of the difference between the velocity of the wheel and that of the impact.

† Sect. III. Prop. IV. Cor. 2.

Fig. LXXIII.

Let the weight of a column of water equal to the force of the impact, when the wheel is quieſcent, be A, and in order to facilitate the ſolution, it muſt be ſuppoſed that the force of the ſtream acts perpendicularly to the planes on the points $I, I, I,$ at a given diſtance from the axis, the radius $SI = a$, $SD = b$, V the velocity with which the water impinges on the boards I, I, I, y the velocity of the circumference I, I, I; then we have this proportion for determining f the force of the water to turn the wheel when its velocity is y, $\overline{V-0}^2 : \overline{V-y}^2 :: A : f = \frac{A\times\overline{V-y}^2}{V^2}$. Let R be the centre of gyration of the wheel and the weight W taken together, and let G be the wheel's weight, and $SR = r$; then ſhall the inertia of the ſyſtem which reſiſts the communication ‡ of motion to I, I, I, be $\frac{\overline{W+G}\times r^2}{a^2}$, and the force which accelerates the circumference when its velocity $= y$ is $\frac{A}{W+G}\times\frac{\overline{V-y}^2}{V^2}\times\frac{a^2}{r^2}$, but from this muſt be ſubducted the force of the weight W to retard the circumference, which is ‖ $= \frac{Wab}{\overline{W+G}\times r^2}$. The force therefore which accelerates the circumference will be upon the whole $= \frac{A}{W+G}\times\frac{\overline{V-y}^2}{V^2}\times\frac{a^2}{r^2} - \frac{Wab}{\overline{W+G}\times r^2}$, and when the wheel's velocity becomes uniform, this force of acceleration muſt be evaneſcent, which will give $\frac{A\times\overline{V-y}^2}{V^2}\times a - Wb = 0$, and $y = V - V\times\sqrt{\frac{W}{A}\times\frac{b}{a}}$ the velocity required.

‡ Sect. VI. Prop. VIII.

‖ Sect. VI. Prop. III. and VIII.

XXXII.

Every thing elſe remaining as in the laſt propoſition, let the weight w vary, and let it be required to aſſign the weight w, ſo that when the wheel has acquired its uniform velocity, the moment of w may be the greateſt poſſible.

Maclaurin's Fluxions, Vol. II. p. 728.

Let the weight ſought $= x$, and ſince the uniform velocity of the * wheel's circumference $= V - V \times \sqrt{\frac{xb}{Aa}}$, the uniform velocity of the aſcent of x will be $\frac{Vb}{a} - V \times \sqrt{\frac{xb^3}{Aa^3}}$, and the moment of x will be $\frac{Vbx}{a} - V \times \sqrt{\frac{x^3b^3}{Aa^3}}$, the fluxion of which being made $= 0$, we have $\dot{x} - \frac{3\sqrt{xb} \times \dot{x}}{2\sqrt{Aa}} = 0$, when the moment of x is the greateſt poſſible, and $x = \frac{4Aa}{9b}$ the weight required.

* Prop. XXXI.

Cor. 1. Let ſuch a weight Q be ſuſpended from the axle, as will exactly balance the force of the water A acting at the circumference, then will $Q = \frac{a \times A}{b}$; wherefore when the moment of the aſcending weight x is the greateſt poſſible, that weight will be $\frac{4Q}{9}$ or $\frac{4}{9}$ parts of the weight which would, if ſuſpended from the axle, balance the force of the ſtream.

Cor. 2. Since the uniform velocity of the wheel's circum-

* Prop. XXXI. ‖ Supra. cumference is *$V - V \times \sqrt{\frac{Wb}{Aa}}$, and ‖$W = \frac{4Aa}{9b}$ when the moment generated in the aſcending weight is the greateſt, by ſubſtituting $\frac{4Aa}{9b}$ for W, we have the velocity $= V - V \times \sqrt{\frac{4}{9}} = \frac{V}{3}$, that is, the velocity of the wheel's circumference will be $\frac{1}{3}$ the velocity of the ſtream impinging againſt it, when the moment generated in the weight aſcending uniformly is the greateſt poſſible.

Cor. 3. The laſt Cor. ſuggeſts another problem which may be of uſe: having given a weight W to be raiſed by the action of the ſtream of water, the force of which is $= A$, againſt a quieſcent ſurface; let it it be required to aſſign what muſt be the proportion between the radius of the wheel and that of the axle, ſo that the uniform velocity of the aſcending weight may be the greateſt poſſible. The value of V remaining, if the radius of the wheel be put $= x$, and $SD = b$, the uniform velocity of the aſcending weight $= \frac{bV}{x} - V \times \sqrt{\frac{Wb^3}{Ax^3}}$, of which if the fluxion be made $= 0$, it will give $x = \frac{9Wb}{4A}$.

Cor. 4. Every thing elſe remaining, if the velocity with which the water impinges againſt the boards I, I, I, be doubled, the greateſt moment communicated to a weight W aſcending uniformly will be increaſed in the ratio of 8 : 1: for ſince the uniform velocity of the aſ‖ Supra p. 277. cending weight is $\frac{bV}{3a}$, and the weight‖ moved $W = \frac{4Aa}{9b}$, the moment generated in the aſcending weight will be expreſſed by $\frac{4AV}{27}$, and $\frac{4}{27}$ being given, the moment will be ‡ Page 276. proportional to $A \times V$, but ‡ A is as V^2, wherefore the moment communicated to the aſcending weight will be as V^3, or in a triplicate ratio of the velocity wherewith the water impinges on the wheel.

The force which communicates motion to water wheels, and the reſiſtances which are occaſioned by friction, tenacity, and various other cauſes, render the application of the theory of mechanics to practice in theſe caſes extremely difficult. It is probably from this reaſon, that the art of conſtructing machines, which are moved by the force of water, &c. has been almoſt wholly practical; the beſt improvements having been deduced from continued obſervation of the reſults produced in given circumſtances, whereby the gradual correction of error, and varied experience of what is moſt effectual, have ſupplied the place of a more perfect inveſtigation from the laws of motion.

This ſeems to be the beſt method, as far as regards the practical conſtruction of theſe machines, the nature of the caſe will admit of; for although there may be two roads leading ultimately to the ſame truths, i. e. a direct inveſtigation from the laws of motion, and long continued obſervation, independent of theory; the latter is frequently the moſt eaſy and moſt eligible, although leſs direct and leſs ſcientific; the former being inacceſſible to thoſe who poſſeſs the elementary parts of mechanics only. It is in vain to attempt the application of the theory of mechanics to the motion of bodies, except every cauſe which can ſenſibly influence the moving power and the reſiſtance to motion be taken into account: if any of theſe be omitted, error and inconſiſtency in the concluſions deduced muſt be the conſequence. It was at one time ſuppoſed from this inadequate application of the theory, that the ſame laws of motion would not extend to all branches of mechanics, but that different principles were to be accommodated to different kinds of motion. If this were truly the caſe, the ſcience of mechanics would fall ſhort of that ſuperior excellence and extent, which it is generally allowed to poſſeſs. For it is probable, that there is no kind of motion, whether of the moſt ſimple or complicated nature, but what may be referred to * three eaſy and obvious propoſitions, the truth of which it is impoſſible to doubt: and if we are not enabled to inveſtigate the effects from the data in all caſes, the deficiency muſt not be imputed to the ſcience of mechanics, but to the want of methods of applying mathematics to it.

This may be illuſtrated by an example in order to ſhew that the motion communicated to water wheels, however complicated the data may be, is equally referable to the laws of motion, with the effects of the moſt uncompounded force. If a ſtream of water falls perpendicularly on a plane ſurface, the moving force ariſing from the impact only is equal to the weight of a column of water, the baſe of which is the ſurface † upon which the water impinges,

* Laws of motion.

† Sect. V. Prop. I.

Fig. LXXIII.

and altitude that through which a body muſt fall from reſt to acquire the velocity of impact. If the inclination of the ſtream to the ſurface ſhould be changed, the force exerted in a direction perpendicular to the plane will be diminiſhed in a duplicate ratio of the radius to the ſine of inclination, the ſurface on which the water impinges remaining. Now when water falls on the boards of a water wheel, the direction of the ſtream makes different angles with the planes of thoſe boards; for ſince the particles of water deſcend in curve lines, they will ſtrike any plane ſurface in the direction of a tangent to the curve on the point of impact. Moreover, the water will ſtrike the higher boards *T*, *T*, with a leſs velocity and in a direction more inclined to their planes, than when it impinges on the boards *I*, *I*: it is alſo to be conſidered, that the ſtream will impinge on the boards at different diſtances from the axis of motion: all which circumſtances muſt be taken into account to find the force which tends to communicate motion to the wheel when quieſcent; and when motion has been communicated, the force of the ſtream to turn the wheel will be determined as in p. 276. But this is not the only conſideration which affects the moving force; the force hitherto conſidered has been ſuppoſed to proceed from the impact of the particles only; in which caſe each particle, after it has ſtruck the board, is imagined to be of no other effect in communicating motion; but this is not wholly the caſe: for after the particle has impinged on the board, it will continue ſome time to operate by its weight, and this time will be longer or ſhorter according to the different conſtructions of the wheel. In the overſhot wheel, the continuance of the preſſure ariſing from the weight of the water, will be longer than in the underſhot, the force which ariſes from the impact of the water being nearly the ſame in each caſe. The whole moving force therefore will conſiſt of the impact determinable as above, and of the weight of the water deſcending along with the circumference and communicating additional motion to it: this entire moving force being determined either by theory or experiment may be denoted by *A*. After the moving force which impels the circumference has been determined, the reſiſtance to this force muſt be found; for on the proportion between the moving force and the reſiſtance, the acceleration of the machine will depend. This reſiſtance is of various kinds: 1. That of inertia: let the centre of gyration of the wheel $= R$, $SR = r$, $SI = a$, the wheel's weight $= G$, then will the inertia of the wheel which reſiſts the communication of motion to the circumference,* $= \frac{W\times r^2}{a^2}$: in the ſame manner, the inertia of any parts of the ſyſtem may be obtained from having given the weights, figures, and diſtances from the axis by prop. XII. 2. If the machine is of that kind which raiſes weights, ſuch for inſtance as water, the weight raiſed, allowing for its mechanical effect on the point the acceleration of which is ſought, muſt be ſubducted from the moving force before found; and this will be a conſtant quantity. There are other reſiſtances alſo homogeneal to weight, i. e. thoſe of friction and tenacity, &c. which are variable in ſome ratio of the machine's velocity: and in order to proceed with the inveſtigation, the exact quantity of weight which the friction is equal to, when the wheel moves with a given velocity, muſt be known, and the variation of the reſiſtances in reſpect of the velocities, which circumſtances muſt be determined by experiment. If the force equivalent to the friction, &c. be ſubducted from the moving force, the remainder will give the moving power by which the circumference is impelled upon the whole; this being divided by the inertia of the maſs moved, will give the force which accelerates the circumference. The notation of the laſt propoſition therefore remaining, if, when the wheel's velocity is V, the friction is equal to the weight Q applied at the circumference, and varies in the nth power of the velocity, we ſhall have the force which accelerates the circumference $\frac{A}{W+G}\times\frac{\overline{V-y}^2}{V^2}\times\frac{a^2}{r^2}-\frac{Wab}{\overline{W+G}\times r^2}-\frac{Qy^n\times a^2}{V^n\times\overline{W+G}\times r^2}$; wherefore if x be the ſpace which has been deſcribed by the circumference when the velocity is y, and l is the ſpace through which bodies are impelled by the force of gravity in one ſecond, the principles of acceleration ‡ give us $\frac{y\dot{y}}{2l}=\overline{\frac{A}{W+G}\times\frac{\overline{V-y}^2}{V^2}\times\frac{a^2}{r^2}-\frac{Wab}{\overline{W+G}\times r^2}-\frac{Qy^n\times a^2}{V^n\times\overline{W+G}\times r^2}}\times\dot{x}$, from which if y be determined in terms of x and conſtant quantities, the velocity communicated to the wheel will be known. It is only by taking in all theſe circumſtances into the account, which can produce coincidence between theory and matter of fact.

* Sect. VI. Prop. VIII.

‡ Sect. III, Prop. V.

‡ Prop. XXXII. Cor. 2.

In Mr. Parent's ‡ propoſition above demonſtrated, the effects of friction are not conſidered, nor is the time in which the uniform motion is produced taken into account: it is not therefore to be wondered, that if this propoſition be referred to experiments made on machines, wherein friction bears a conſiderable proportion to the moving force, the reſults ſhall be different from thoſe demonſtrated from different data. The motion of machines impelled by the force of water or wind very ſoon becomes uniform; the reaſon of which is, that the friction of the parts vary in ſome direct ratio of the velocity: for were friction entirely removed, the motion of the machine would be continually accelerated, and would not acquire its uniform velocity in any finite time. If the exact laws according to which the friction of the machine varied in reſpect of the velocity were known, together with the abſolute quantity of friction correſponding to a given velocity, the uniform or greateſt velocity acquirable by the machine, and the finite time of acquiring it might be aſcertained; but it will appear from the following propoſition, that when friction is not taken into account, the time of acquiring the uniform velocity, ſuch as is referred to in Mr. Parent'spropoſition, is infinite.

XXXIII.

Every thing remaining as in the XXXIſt prop. let it be required to aſſign the time during which the wheel A B C is accelerated before it acquires a given velocity, effects of friction, coheſion, &c. not being conſidered.

The notation remaining, let x be the ſpace which has been deſcribed by any point of the wheel's circumference when the velocity generated has become $= y$: the force

‡ Page 276.

which ‡ accelerates the wheel's circumference $= \frac{A}{W+G} \times \frac{\overline{V-y}^{\,2}}{V^2} \times \frac{a^2}{r^2} - \frac{Wab}{\overline{W+G} \times r^2}$, for brevity let $\frac{A}{W+G} \times \frac{a^2}{r^2}$ $=$

$= F$, and $\frac{Wab}{\overline{W+G} \times r^2} = K$; we have therefore the force which accelerates the circumference $= \frac{\overline{V-y}^2 \times F}{V^2} - K$: let z be the ſpace through which a body falls freely from reſt to acquire the velocity y, then * will $z = \frac{y^2}{4l}$, and the principles of acceleration give us this equation $\frac{y\dot{y}}{2l} = \frac{\overline{\overline{V-y}^2 \times F - V^2K} \times \dot{x}}{V^2}$, and $\dot{x} = \frac{V^2\dot{y}y}{2l \times \overline{\overline{V-y}^2 \times F - V^2K}}$, and the fluxion of the time $\dot{t} = \frac{\dot{x}}{y} = \frac{V^2\dot{y}}{2l \times \overline{\overline{V-y}^2 \times F - V^2K}}$; wherefore taking the fluents, the time itſelf $= \frac{V}{\sqrt{4FKl^2}} \times$ log. $\frac{V\sqrt{K} + \overline{V-y}\sqrt{F}}{\sqrt{\overline{V-y}^2 \times F - V^2}\sqrt{K}}$: but this ſhould vaniſh when $y = 0$; wherefore the entire fluent or the time required t

$$= \frac{V}{\sqrt{4l^2FK}} \times \text{log.}\ \frac{V\sqrt{K} + \overline{V-y}\sqrt{F}}{\sqrt{\overline{V-y}^2 \times F - V^2K}} \times \frac{\sqrt{F-K}}{\sqrt{F} + \sqrt{K}}.$$

In order to reduce this, let $\frac{\sqrt{F-K}}{\sqrt{F} + \sqrt{K}} = n$, $V - y = z$, and $e =$ the number, the hyperbolic log. of which $= 1$, and let $\sqrt{\frac{4l^2FKt^2}{V^2}} = p$; this will give $\frac{nV\sqrt{K} + nz\sqrt{F}}{\sqrt{z^2F - V^2K}} = e^p$, and ſquaring the whole $\frac{n^2V^2K + 2n^2Vz\sqrt{FK} + n^2z^2F}{Fz^2 - V^2K} = e^{2p}$, and $Fz^2 - V^2K = e^{-2p} \times \overline{n^2V^2K + 2n^2Vz\sqrt{FK}e^{-2p} + n^2z^2F} \times e^{-2p}$: this being reſolved gives

$$z = V - y = \frac{n^2V\sqrt{FK}e^{-2p}}{F - n^2Fe^{-2p}} + \sqrt{\frac{n^4V^2FKe^{-4p}}{\overline{F - n^2Fe^{-2p}}^2} + \frac{V^2K + e^{-2p} \times n^2V^2K}{F - n^2Fe^{-2p}}};$$

which being ſubtracted from V, leaves $y =$ the velocity of the circumference generated in the time t from reſt.

* Sect. III, Prop. V.

Cor. As the time t increaſes, the wheel's velocity increaſes, and if it has any limit by making t infinite, the ultimate value of y will give that limit: here ſince $p = \sqrt{\frac{4l^2t^2FK}{V^2}}$, it is manifeſt, that if t be infinite, p will be alſo infinite, and e^{-2p} evaneſcent; wherefore all the terms in the preceding expreſſion which are multiplied into e^{-2p} will become $= 0$: this will give the ultimate value of $z = V - y =$ to $V\sqrt{\frac{K}{F}}$, and y the limit of the velocity ſought will be $= V - V\sqrt{\frac{K}{F}}$; but $K = \frac{Wab}{\overline{G+W} \times r^2}$, and $F = \frac{A \times a^2}{\overline{G+W} \times r^2}$, wherefore $\frac{K}{F} = \frac{Wb}{Aa}$, and y the greateſt velocity ‡ $= V - V\sqrt{\frac{Wb}{Aa}}$: this was otherwiſe ſhewn to be the uniform or greateſt velocity of motion which could be acquired by the circumference, which it appears from this ſolution it never can arrive at in any finite time.

‡ Prop. XXXI.

The chief propoſition upon which the eſtimation of the velocities generated in a ſyſtem of bodies revolving round a fixed axis depends, is that by which the inertia of the various parts of the ſyſtem is determined: for after it has been ſhewn what maſs, when acted on immediately by the moving force, is equivalent in its inertia to that which is exerted by the parts of the ſyſtem revolving at their reſpective diſtances, the acceleration of the point at which the moving force is applied becomes known: it has been ſhewn, that the inertia exerted by the particles, which revolve at different diſtances † from the axis of motion, are in a ratio compounded of the direct duplicate ratio of their velocities or diſtances from the axis, and the direct ratio of the quantities of matter contained in them; from which principle the figures and weights of the various parts which compoſe the ſyſtem being known, its revolving motion generated by a given moving force can be eſtimated: but theſe propoſitions cannot be generally applied to the determination of the inertia exerted by bodies againſt the action of moving forces, becauſe the conditions on which they are demonſtrated are particular in belonging to rotatory motion, or others of the ſame kind; the cotemporary

† Sect. VI. Prop. I. & II.

rary velocities of any given points in the ſyſtem are in a conſtant ratio from the firſt inſtant of motion; moreover, the moving forces, as well as the inertia of the various parts are conſtant, becauſe in revolving bodies the diſtance from the axis of each particle, as well as the diſtance from the axis at which the moving force is applied, is invariable: and it follows, that in other caſes, where theſe conditions exiſt not, the laws which have been demonſtrated for the eſtimation of the acceleration of revolving bodies cannot be applied; other rules therefore will be neceſſary for this purpoſe.

XXXIV.

Fig. LXXIV.

Let AFC, BGC be two curves, the planes of which are vertical, and let a line ACB be ſtretched over a fixed pully C by two given weights A and B, of which B preponderates againſt A, and deſcends along the curve CGB: it is required to aſſign the velocity of B when it has deſcended through a given perpendicular altitude, and A has aſcended through a given altitude in the ſame time.

Let BL be the perpendicular altitude through which B has deſcended, and let AH be the perpendicular altitude thro' which A has aſcended in the ſame time, and let $BL = y$, $AH = q$. Suppoſe the weight B to deſcribe in its deſcent the evaneſcent arc bB, and during the ſame time let the other weight A riſe through the arc aA. Through b and A draw bo and Al parallel to the horizon: alſo, through b draw bn perpendicular to CB, and Am perpendicular to Ca, and let $CB = x$, $Bn = \dot{x} = Am$, $Bo = \dot{y}$, $Bb = \dot{z}$, $Aa = \dot{r}$, and $al = \dot{q}$; and let z be the ſpace through which a body muſt fall freely by the acceleration of gravity from reſt ſo as to acquire the velocity of B, while it is deſcribing the elementary ſpace bB.

The force which impels B in the direction bB is $= \frac{B\dot{y}}{\dot{s}}$, and ſince A acts in oppoſition to this force, its effects when referred to the direction bB muſt be ſubducted from $\frac{B\dot{y}}{\dot{s}}$ in order to obtain the force which upon the whole impels B in the direction bB: the force of A in the direction $BC = \frac{A\dot{q}}{\dot{x}}$, which being reſolved into the direction bB becomes $= \frac{A\dot{q}}{\dot{s}}$; wherefore the whole force by which B is urged in the direction bB will $= \frac{B\dot{y}}{\dot{s}} - \frac{A\dot{q}}{\dot{s}}$.

Alſo, the force with which B impels A in the direction $AC = \frac{B\dot{y}}{\dot{x}}$, and that of A in the ſame direction $= \frac{A\dot{q}}{\dot{x}}$, wherefore the entire force which urges A in the direction $AC = \frac{B\dot{y} - A\dot{q}}{\dot{x}}$, and this force being reſolved into the direction aA becomes $\frac{B\dot{y} - A\dot{q}}{\dot{r}}$: with this force therefore B impels A in the direction aA. It is next to be enquired, what is the maſs moved by B, or the inertia which is oppoſed to the communication of motion to B: the inertia of B oppoſed by its own maſs is B, but as the bodies A and B move with different velocities, the inertia oppoſed by A will not be equivalent to the maſs A, for reaſons before aſſigned. In order to determine an equivalent maſs, which, being accumulated in B when A is removed, will reſiſt the communication of motion in the ſame manner, as the maſs A aſcending by the force of B, let B be firſt ſuppoſed without inertia, and to poſſeſs weight only: then ſince the force which impels A in the direction aA is $\frac{B\dot{y} - A\dot{q}}{\dot{r}}$, and the maſs moved A, B being ſuppoſed without inertia, the force which accelerates A will be denoted by $\frac{B\dot{y} - A\dot{q}}{\dot{r}A}$, becauſe A is now the only maſs of matter moved. But A deſcribes the ſpace $\dot{r}$ in the ſame particle of time in which B deſcribes bB or

$\dot{s}$; and ſince when the fluxion of the time is given, the force which accelerates bodies deſcribing different ſpaces will be [illegible] as the ſecond fluxions of thoſe ſpaces, we have the force which accelerates the body $B = \frac{\overline{B\dot{y} - A\dot{q}}}{\dot{r}A} \times \frac{\ddot{s}}{\ddot{r}}$, from hence we ſhall obtain the maſs, which if placed at *B* would exert a reſiſtance or inertia equivalent to *A*: for ſince in this caſe, the moving force and inertia are each applied at the ſame point, and referred to the ſame direction, the quantity of matter * moved will be equal to the moving force divided by the accelerating force: and ſince the force which impels *B* is $\frac{B\dot{y} - A\dot{q}}{\dot{s}}$, and the force which accelerates $B = \frac{\overline{B\dot{y} - A\dot{q}}}{A\dot{r}} \times \frac{\ddot{s}}{\ddot{r}}$, we have the maſs moved at *B* or the inertia equivalent to that of $A = \frac{A\dot{r}\ddot{r}}{\dot{s}\ddot{s}}$.

From hence a ſolution to the problem above deſcribed immediately follows; for let z be the ſpace through which a body muſt fall freely to acquire the velocity in *B*, then the force which impels *B* in the direction *bB* is $\frac{B\dot{y} - A\dot{q}}{\dot{s}}$, and the inertia of *B* is *B*, that of *A* equivalent to a maſs $\frac{A\dot{r}\ddot{r}}{\dot{s}\ddot{s}}$ when accumulated in *B*, *A* itſelf being taken away, the whole maſs moved therefore by *B* will be equivalent to $B + \frac{A\dot{r}\ddot{r}}{\dot{s}\ddot{s}} = \frac{B\dot{s}\ddot{s} + A\dot{r}\ddot{r}}{\dot{s}\ddot{s}}$, and the force which accelerates $B = \frac{\ddot{s} \times \overline{B\dot{y} - A\dot{q}}}{B\dot{s}\ddot{s} + A\dot{r}\ddot{r}}$: and ſince $\dot{s}$ is the fluxion of the ſpace deſcribed by *B*, the principles of acceleration give us $\dot{z} = \frac{\dot{s}\ddot{s} \times \overline{B\dot{y} - A\dot{q}}}{B\dot{s}\ddot{s} + A\dot{r}\ddot{r}}$, and $z =$ the fluent of $\dot{s}\ddot{s} \times \frac{\overline{B\dot{y} - A\dot{q}}}{B\dot{s}\ddot{s} + A\dot{r}\ddot{r}}$.

Cor. 1. Let *AFC* become a ſtraight line perpendicular to the horizon, then will $\dot{r} = \dot{x} = \dot{q}$, it follows therefore, that $\dot{z} = \frac{\dot{s}\ddot{s} \times \overline{B\dot{y} - A\dot{x}}}{B\dot{s}\ddot{s} + A\dot{x}\ddot{x}}$.

Cor,

* Sect. I. Prop. IX.

Fig LXXV

Cor. 2. Let CFA be a ſtraight line perpendicular to the horizon, and let the curve CGB be a loragithmic ſpiral, the ordinates of which interſect the curve at an angle the coſine of which is to radius in the proportion of c to r; then the notation remaining, $r\dot{x} = c\dot{s}$ by the property of the curve, and $r\ddot{x} = c\ddot{s}$, wherefore $\frac{\dot{s}\ddot{s}}{B\dot{s}\ddot{s} + A\dot{x}\ddot{x}} = \frac{r^2}{r^2B + c^2A}$, and $z = \frac{r^2 \times \overline{By - Ax}}{Br^2 + c^2A}$, and if $l = 193$ inches, the velocity acquired by $B = \sqrt{\frac{4lr^2 \times \overline{By - Ax}}{Br^2 + c^2A}}$.

Cor. 3. The laws obſerved by bodies which revolve round a fixed axis of motion are particular caſes of this general propoſition, and may be inferred from it; thus, let LMN repreſent a wheel and axle moveable round an horizontal axis of motion; and ſuppoſe that a weight A is applied to the circumference of the axle to be raiſed by the weight B applied to the wheel's circumference: and let the radius $SL = r$, and $SM = c$, the force whereby B endeavours to deſcend $= B - \frac{Ac}{r} = \frac{Br - Ac}{r}$: let any ſpace deſcribed by B during its deſcent from reſt $= y$, and the ſpace deſcribed by A in its aſcent during the ſame time $= x$; then will $rx = cy$, wherefore $r\dot{x} = c\dot{y}$ and $r\ddot{x} = c\ddot{y}$; and ſince the inertia oppoſed by B to its own motion is B, and the inertia oppoſed by A to the motion of B is by the preceding ſolution $= \frac{A\dot{x}\ddot{x}}{\dot{y}\ddot{y}} = \frac{Ac^2}{r^2}$, the whole inertia oppoſed to the motion of $B = B + \frac{Ac^2}{r^2} = \frac{Br^2 + Ac^2}{r^2}$, and ſince the force which impels $B = \frac{Br - Ac}{r}$, the force which * accelerates the deſcent of B will be $= \frac{Br^2 - Acr}{Br^2 + Ac^2}$. Let z be the ſpace through which a body muſt fall from reſt freely, by the acceleration of gravity, to acquire the velocity of B at the inſtant it has deſcribed the ſpace y; this † will give $\dot{z} = \frac{Br^2\dot{y} - Acr\dot{y}}{Br^2 + Ac^2}$, and $z = \frac{Br^2y - Acry}{Br^2 + Ac^2}$; or ſince $y =$

rx

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* Sect. I. Prop. IX.

† Sect. III. Prop. V.

$\frac{rx}{c}$, $z = \frac{Byr^2 - Ar^2x}{Br^2 + Ac^2}$, which is preciſely the ſame value as is otherwiſe obtained ‡ from prop. xix. the wheel's inertia not being conſidered. In the ſame manner, the inertia of the whole ſyſtem, including that of the wheel, might be inferred from the preceding general ſolution. ‡ Sect. VI.

In the examples which have been given to illuſtrate this propoſition, the ratio of the fluxions $\dot{x} : \dot{y}$ & $\ddot{y} : \ddot{x}$ is conſtant; in that which follows they are variable. Let two equal weights *A*, *A* be faſtened to the extremities of a line which goes over the fixed points *E* and *C*, which are horizontal: when the line *AECA* is ſtretched by the equal weights *A*, *A*, let a body *B* be fixed to the middle point *D*, it will deſcend from reſt at *D* in the direction *DB*, perpendicular to the horizon, at the ſame time elevating the weights *A*, *A*. Suppoſe it were required to aſſign the velocity of the deſcending weight *B* when it has deſcribed any ſpace *DB*. Let $ED = a$, $EB = y$, and $BD = x$: to determine the velocity of *B*, while it deſcribes the evaneſcent ſpace $bB = \dot{x}$. *B* is impelled downward by its own weight diminiſhed by $2A \times \frac{x}{y} = 2A \times \frac{x}{\sqrt{a^2 + x^2}}$ acting in the contrary direction *BD*: the whole force therefore by which *B* tends downwards $= B - \frac{2Ax}{\sqrt{a^2+x^2}}$: the maſs moved by *B* is its own quantity of matter, together with the inertia of the weights *A*, *A* referred to the direction *Bb*; the inertia of $2A$ eſtimated in the direction $Bb = 2A \times \frac{\dot{y}\ddot{y}}{\dot{x}\ddot{x}}$ by the propoſition above demonſtrated, and becauſe $\frac{\dot{y}}{\dot{x}} = \frac{x}{\sqrt{a^2 + x^2}}$, and $\frac{\ddot{y}}{\ddot{x}} = \frac{a^2 + 2x^2}{2x \times \sqrt{a^2 + x^2}}$, the inertia of $2A$ will be $2A \times \frac{\dot{y}\ddot{y}}{\dot{x}\ddot{x}} = A \times \frac{a^2 + 2x^2}{a^2 + x^2}$, and that of $B + 2A$ will $= \frac{B \times \overline{a^2 + x^2} + A \times \overline{a^2 + 2x^2}}{a^2 + x^2}$: let z be the ſpace due to the velocity of *B* while deſcribing *Bb*, and ſince the force which impels $B = \frac{B \times \sqrt{a^2 + x^2} - 2Ax}{\sqrt{a^2 + x^2}}$, we ſhall * have $\dot{z} =$ Fig. LXXXIX.

* Sect. III. Prop. V. Cor. 4.

Ba^2

$\frac{Ba^2\dot{x}+Bx^2\dot{x}-2Ax\dot{x}\sqrt{a^2+x^2}}{Ba^2+Bx^2+Aa^2+2Ax^2}$, and taking the fluents $z = \frac{BAa\times M^\circ}{\sqrt{B+A}\times\overline{B+2A}^{\frac{3}{2}}} + \frac{Bx+2Aa-2A\times\sqrt{a^2+x^2}}{B+2A} + \frac{2A^{\frac{3}{2}}a}{\overline{B+2A}^{\frac{3}{2}}} \times \log. \frac{\sqrt{Aa^2}+\sqrt{\overline{B+2A}\times\overline{a^2+x^2}}}{\sqrt{\overline{B+A}\times a^2+\overline{B+2A}\times x^2}} \times \frac{\sqrt{B+A}}{\sqrt{A}+\sqrt{B+2A}}$: in which M° ſignifies the arc of a circle, the tangent of which is $\sqrt{\frac{B+2A}{B+A}} \times \frac{x}{a}$ to radius $= 1$.

Cor. 1. By inſpecting the value of z, it appears, that when A is $= 0$, or when B is infinite, $z = x$.

Cor. 2. If $B = 2A$, B will deſcend continually, but the velocity acquired will be limited; by making x infinite, $A = 1$, and $B = 2$, $p = 3.14159$, &c. the limit of z will become $\frac{pa}{8\sqrt{3}} - \frac{a\times\log.\sqrt{3}}{4} = a \times .2161$, ſo that the velocity of the deſcending weight B, is always leſs than that which a heavy body acquires by falling freely from reſt through .2161 part of half the ſtring's length.

Cor. 3. If B be in the leaſt greater than $2A$, the value of z, and conſequently the velocity, when x is infinite, will increaſe ſine limite.

Cor. 4. When B is leſs than $2A$, the velocity of B during its deſcent will firſt increaſe and afterwards decreaſe, until the weights become ſtationary: B will then begin to aſcend.

Cor. 5. By making the fluxion of $z = 0$, or $\dot{z} = \frac{Ba^2\dot{x}+Bx^2\dot{x}-2Ax\dot{x}\times\sqrt{a^2+x^2}}{Ba^2+Bx^2+Aa^2+2Ax^2} = 0$, and ſolving the equation, we ſhall have the value of x when the velocity of B is the greateſt poſſible in general terms, in which caſe it appears that $x = \frac{Ba}{\sqrt{4A^2-B^2}}$.

SECT.

SECT. VII.

CONTAINING A DESCRIPTION OF EXPERIMENTS ON THE RECTILINEAR MOTION OF BODIES, WHICH ARE ACTED ON BY CONSTANT FORCES.

MECHANICAL experiments are of two ſorts; the one relating to the quieſcence of bodies, and the other to their motion. Among the former are included thoſe which demonſtrate, or rather make evident to the ſenſes, the equilibrium of the mechanic powers, and the correſponding proportions of the weights ſuſtained to the forces which ſuſtain them, the properties of the centre of gravity, the compoſition and reſolution of forces, &c.: by the latter kind of experiments are ſhewn the laws of colliſion, of acceleration, and the various effects of forces, which communicate motion to bodies.

Concerning mechanical experiments, it is obſervable, that thoſe wherein an equilibrium is formed, will frequently appear coincident with the theory, although conſiderable errors are committed in conſtructing them. To exemplify this, let A B repreſent a rod moveable round an horizontal axis of motion, which paſſes through its centre of gravity S. Suppoſe S A to S C in the proportion of 2 : 1; and let any weight W be ſuſpended from A; in order to balance W by a weight acting at C, this weight muſt be accurately = 2 W by the theory: but if, by an error in conſtructing the experiment, inſtead of 2 W, another weight greater or leſs than 2 W be applied at C, an equilibrium will be ſtill produced, provided the friction of the axis be ſufficient to counteract the moving force ariſing from the erroneous weight. In the ſame manner, it appears in other caſes, that from the effects of friction, tenacity, &c. experiments relating to the equilibrium of forces will derive an appearance of greater exactneſs, than would be obſerved in them were friction, &c. wholly removed. The caſe is different in experiments concerning the motion of bodies, in which, whatever care be taken to render the proportion of the forces and of the weights moved, ſuch as is required by the theory, the interference of friction and other reſiſtances, which contribute to render the experiments on the equilibrium Fig. LXXVI.

of forces apparently more perfect than they really are, cauſes the motion of bodies, which are the objects of experiments, to differ from the theory, with which it would preciſely coincide were thoſe obſtacles removed.

According to the greater or leſs degree of exactneſs, experiments on the motion of bodies are required to correſpond with the theory, more or leſs attention is neceſſary in removing the impediments above deſcribed, as well as others which will be hereafter conſidered: if the experiments are deſigned only to aſſiſt the imagination by ſubſtituting ſenſible objects, inſtead of abſtract and ideal quantities, an apparent agreement between the theory and experiment may be ſufficient to anſwer this purpoſe, although it may be produced from an erroneous conſtruction; but it muſt be allowed, that experiments of this kind are extremely defective, and entirely inſufficient to impreſs the mind with that ſatisfactory conviction, which always attends the obſervation of experiments accurately made.

It may perhaps be required to aſſign what degree of preciſion is neceſſary in experiments on the motion of bodies ſince ma-ing them. To exemplify this, let A B repreſent a rod moveable round an horizontal axis of motion, which paſſes through its centre of gravity S. Suppoſe S A to S C in the proportion of 2 : 1; and let any weight W be ſuſpended from A; in order to balance W by a weight acting at C, this weight muſt be accurately = 2 W by the theory: but if, by an error in conſtructing the experiment, inſtead of 2 W, another weight greater or leſs than 2 W be applied at C, an equilibrium will be ſtill produced, provided the friction of the axis be ſufficient to counteract the moving force ariſing from the erroneous weight. In the ſame manner, it appears in other caſes, that from the effects of friction, tenacity, &c. experiments relating to the equilibrium of forces will derive an appearance of greater exactneſs, than would be obſerved in them were friction, &c. wholly removed. The caſe is different in experiments concerning the motion of bodies, in which, whatever care be taken to render the proportion of the forces and of the weights moved, ſuch as is required by the theory, the interference of friction and other reſiſtances, which contribute to render the experiments on the equilibrium Fig. LXXVI.

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It may perhaps be required to aſſign what degree of preciſion is neceſſary in experiments on the motion of bodies ſince maperiments on bodies, which deſcend through conſiderable altitudes, and if the altitudes be diminiſhed, the times of motion will be ſo ſmall as to render the obſervation of them very precarious and unſatisfactory.

If to remedy this inconvenience bodies be cauſed to deſcend along inclined planes, according to the experiments of the celebrated author* of this theory, by varying the proportion of the planes' heights to their lengths, the force of acceleration may be diminiſhed in any ratio, ſo that the deſcending bodies ſhall move ſufficiently ſlow to allow of the times of motion from reſt being accurately obſerved; and the effects of the air's reſiſtance to bodies moving with theſe ſmall velocities will be abſolutely inſenſible: the principal difficulty however which here occurs, ariſes from the rotation of the deſcending bodies, which cannot be prevented without increaſing their friction far beyond what the experiment will allow of.

* Galilæo.

Fig. LXXVII. To conſider this a little further: Bodies thus deſcending along inclined planes by rotation, with an uniformly accelerated motion, may be either ſpherical or cylindrical. Let A B C repreſent an inclined plane, the height of which is to its length as 1 : 9.2; then if a body deſcends from reſt by ſliding along this plane, and the ſurfaces of the plane, and of the deſcending body be perfectly free from friction, ſince the force of acceleration is $\frac{1}{9.2}$ part of acceleration of gravity, the ſpace deſcribed in one ſecond from * reſt $= \frac{193}{9.2} = 21$ inches: but if a ſphere deſcends by rolling along the ſame inclined plane, the ſpace deſcribed in one ſecond † from reſt will be only $\frac{21 \times 5}{7} = 15$ inches; and if a cylinder ſhould roll down the plane inſtead of a ſphere, the ſpace deſcribed in one ‡ ſecond would be no more than 14 inches, inſtead of 21 inches, the ſpace which would be deſcribed from reſt in one ſecond, by a cylinder or any other body ſliding along the plane: it may however be obſerved, that in each of theſe deſcents, the body will be uniformly accelerated, and conſequently the ſpaces deſcribed in a duplicate ratio of the times of motion eſtimated from the body's quieſcence.

But in theſe methods of making experiments on the accelerated motion of bodies,

* Sect. III. Prop. IV.

† Sect. VI. Prop. XIV.

‡ Sect. VI. Prop. XIV. Cor. 2.

dies, there are no means ſeparating the maſs moved from the moving force; we cannot therefore apply different forces to move the ſame quantity of matter on a given plane, or the ſame force to different quantities of matter. Moreover, the accelerating force being conſtant and inſeparable from the body moved, its velocity will be continually accelerated, ſo as to render the obſervation of the velocity acquired at any given inſtant impoſſible. It may be therefore not altogether uſeleſs to deſcribe ſuch an inſtrument as will ſubject to experimental examination, the properties of the five mechanical quantities, i. e. the quantity of matter moved, the conſtant force which moves it, the ſpace deſcribed from reſt, the time of deſcription, and the velocity acquired.

Of the maſs moved.

1. In order to obſerve the effects of the moving force, which is the object of any experiment, the interference of all other forces ſhould be prevented: the quantity of matter moved therefore, conſidering it before any impelling force has been applied, ſhould be without weight. Although it be impoſſible to abſtract the natural gravity or weight from any ſubſtance whatever, yet the weight may be ſo coun-

teracted as to be of no ſenſible effect in experiments. Thus in the inſtrument Fig. LXXVIII. conſtructed to illuſtrate this ſubject experimentally, A, B repreſent two equal weights affixed to the extremities of a very fine and flexible ſilk line: this line is ſtretched over a wheel or fixed pully a b c d, moveable round an horizontal axis: the two weights A, B being preciſely equal and acting againſt each other, remain in equilibrio; and when the leaſt weight is ſuperadded to either (ſetting aſide the effects of friction) it will preponderate. When A, B are ſet in motion, by the action of any weight m, the ſum A + B + m would conſtitute the whole maſs moved, but for the inertia of the materials which muſt neceſſarily be uſed in the communication of motion: theſe materials conſiſt of 1. the wheel a b c d over which the line ſuſtaining A and B paſſes. 2. The four friction wheels on which the axle of the wheel a b c d reſts: the uſe of theſe wheels is to prevent the loſs of motion, which would be occaſioned by the friction of the axle if it revolved on an immoveable ſurface. 3. The line by which the bodies A and B are connected ſo as when ſet in motion to

move

move with equal velocities. The weight and inertia of the line are too ſmall to have ſenſible effect on the experiments, as will appear in a ſubſequent note: but the inertia of the other materials juſt mentioned conſtitute a conſiderable proportion of the maſs moved, and muſt be taken into account. Since when A and B are put in motion, they muſt neceſſarily move with a velocity equal to that of the circumference of the wheel a b c d, to which the line is applied; it follows, that if the whole maſs of the wheels were accumulated in this circumference, its inertia would be truly eſtimated by the quantity of matter moved: but ſince the parts of the wheels move with different velocities, their effects in reſiſting the communication of motion to A and B by their inertia will be different; thoſe parts which are furtheſt from the axis reſiſting more than thoſe which revolve nearer, in a duplicate proportion * of thoſe diſtances. If the figures of the wheels were regular, from knowing their † weights and figures, the diſtances of their centres of gyration from their axes of motion would become known, and conſequently an equivalent weight, which being accumulated uniformly in the ‡ cir-

cum-

* Sect. VI. Prop. II.

† Sect. VI. Prop. VIII.

‡ Prop. VIII. and XXIII.

cumference a b c d, would exert an inertia equal to that of the wheels in their conſtructed form. But as the figures are wholly irregular, recourſe muſt be had to experiment in order to aſſign what equivalent quantity of matter, which being accumulated uniformly in the circumference of the wheel a b c d, would reſiſt the communication of motion to A in the ſame manner as the wheels themſelves.

In order to aſcertain the inertia * of the wheel a b c d, with that of the friction of the wheels, the weights A, B being removed, the following experiment was made.

A weight of 30 grains was affixed to a ſilk line (the weight of which was not ſo much as $\frac{1}{4}$ of a grain, and conſequently too inconſiderable to have ſenſible effect in the experiment) this line being wound round the wheel a b c d, the weight 30 grains by deſcending from reſt communicated motion to the wheel, and by many trials was obſerved to deſcribe a ſpace of about $38\frac{1}{2}$ inches in 3 ſeconds. From theſe data the equivalent maſs or inertia of the wheels will be known from the rule contained ‡ in prop. XIII. ſect. VI. Applying this rule to the preſent

* Sect. VI. Prop. XIII.

‡ Page 226.

ſent caſe, we have $p = 30$ grains, $t = 3$ ſeconds; $l = 193$ inches, $s = 38.5$ inches, and the inertia ſought $= \frac{p \times t^2 l}{s} - p = \frac{30 \times 9 \times 193}{38.5} - 30 = 1323$ grains, or $2\frac{3}{4}$ ounces. This is the inertia equivalent to that of the wheel a b c d, and the friction wheels together; for the rule to which the preceding experiment was referred, extends to the eſtimation of the inertia of the maſs contained in all the wheels.

Fig. LXXVIII.

The reſiſtance to motion therefore ariſing from the wheel's inertia will be the ſame as if they were abſolutely removed, and a maſs of $2\frac{3}{4}$ ounces were uniformly accumulated in the circumference of the wheel a b c d. This being premiſed, let the boxes A and B be replaced, being ſuſpended by the ſilk line over the wheel or pully a b c d, and balancing each other: ſuppoſe that any weight m be added to A ſo that it ſhall deſcend; the exact quantity of matter moved, during the deſcent of the weight A, will be aſcertained, for the whole maſs will be $A + B + m + 2\frac{3}{4}$ oz.

To proceed in deſcribing the conſtruction of the enſuing experiments. In order to avoid troubleſome computations in adjuſting the quantities of matter moved and the moving forces, ſome determinate weight of convenient magnitude may be aſſumed as a ſtandard, to which all the others are referred. This ſtandard weight in the ſubſequent experiments is $\frac{1}{4}$ of an ounce, and is repreſented by the letter m. The inertia of the wheels being therefore $= 2\frac{3}{4}$ ounces will be denoted by 11 m. A and B are two boxes conſtructed ſo as to contain different quantities of matter, according as the experiment may require them to be varied: the weight of each box, including the hook to which it is ſuſpended $= 1\frac{1}{2}$ oz. or according to the preceding eſtimation, the weight of each box will be denoted by 6 m; theſe boxes contain ſuch weights as are repreſented by C, each of which weighs an ounce, ſo as to be equivalent to 4 m: other weights of $\frac{1}{2}$ an oz. $= 2$ m, $\frac{1}{4} =$ m, and aliquot parts of m, Fig. LXXIX.

ſuch as $\frac{1}{2}$ m, $\frac{1}{4}$ m, may be alſo included in the boxes, according to the conditions of the different experiments hereafter deſcribed.

If $4\frac{3}{4}$ oz. or 19 m, be included in either box, this with the weight of the box itſelf will be 25 m; ſo that when the weights A and B, each being 25 m, are balanced in the manner above repreſented, their whole maſs will be 50 m, which being added to the inertia of the wheels 11 m, the ſum will be 61 m. Moreover, three circular weights, ſuch as that which is repreſented at fig. LXXX. are conſtructed; each of which = $\frac{1}{4}$ oz. or m: if one of theſe be added to A and one to B, the whole maſs will now become 63 m, perfectly in equilibrio, and moveable by the leaſt weight added to either (ſetting aſide the effects of friction) in the ſame manner preciſely as ifthe ſame weight or force were applied to communicate motion to the maſs 63 m, exiſting in free ſpace and without gravity.

Of the moving force.

2. Since the natural weight or gravity of any given ſubſtance is conſtant, and the exact quantity of it eaſily eſtimated, it

will be convenient in the ſubſequent experiments to apply a weight to the maſs A as a moving force: thus, when the ſyſtem conſiſts of a maſs = 63 m, according to the preceding deſcription, the whole being perfectly balanced, let a weight $\frac{1}{4}$ oz. or m, ſuch as is repreſented in fig. LXXX. be applied on the maſs A, this will communicate motion to the whole ſyſtem: by adding a quantity of matter m to the former maſs 63 m, the whole quantity of matter moved will now become 64 m, and the moving force being = m, this will give the force which accelerates the * deſcent of A = $\frac{m}{64\,m}$ or $\frac{1}{64}$ part of the accelerating force by which the bodies deſcend freely towards the earth's ſurface.

* Sect. I. Prop. IX.

By the preceding conſtruction the moving force may be altered without altering the maſs moved; for ſuppoſe the three weights m, two of which are placed on A, and one on B to be removed, then will A balance B. If the weights 3 m be all placed on A, the moving force will now become 3 m, and the maſs moved 64 m as before, and the force which accelerates the deſcent of A $= \frac{3\,m}{64\,m} = \frac{3}{64}$ part of the force by which gravity accelerates bodies in their free deſcent to the earth's ſurface.

Suppoſe it were required to make the moving force 2 m, the maſs moved continuing the ſame. In order to effect this, let the three weights, each of which = m, be removed; A and B will balance each other; and the whole maſs will be 61 m: Fig. LXXXI. let $\frac{1}{2}$ m be added to A, and $\frac{1}{2}$ m to B, the equilibrium will ſtill be preſerved, and the maſs moved will be 62 m; now let 2 m be added to A, the moving force will be 2 m, and the maſs moved 64 m as before; wherefore the force of acceleration $= \frac{1}{32}$ part of the acceleration of gravity. Theſe alterations in the moving force may be made with great eaſe and convenience in the more obvious and elementary experiments, there being no neceſſity for altering the contents of the boxes A and B: but the proportion and abſolute quantities of the moving force and maſs moved may be of any aſſigned magnitude, according to the conditions of the propoſition to be illuſtrated: thus, let any number of the weights m, for example 11 m, be included in A, and 8 m in B, then will the moving force be 3 m, and the maſs moved 14 m + 17 m + 11 m, (becauſe 11 m is the inertia of the wheels, and each of the boxes = 6 m) making altogether 42 m; wherefore the accelerating force in this caſe is that part of the acceleration of gravity which is expreſſed by the fraction $\frac{3\text{ m}}{42\text{ m}} = \frac{1}{14}$.

3. The moving force and the maſs moved, and conſequently the force which accelerates the bodies, having been conſtructed; the method of eſtimating practically the ſpace deſcribed from quieſcence is next to be conſidered. The body A deſcends in a vertical line; and a ſcale about 64 inches in length graduated into inches and tenths of an inch is adjuſted vertical, and ſo placed that the deſcending weight A may fall in the middle of a ſquare ſtage, fixed to receive it at the end of the deſcent: the beginning of the deſcent is eſtimated from Q on the ſcale, when the bottom of the box

Of the ſpace deſcribed,

Fig. LXXVIII,

A is on a level with O. The deſcent of A is terminated when the bottom of the box ſtrikes the ſtage, which may be fixed at different diſtances from the point O, ſo that by altering the poſition of the ſtage, the ſpace deſcribed from quieſcence may be of any given magnitude leſs than 64 inches.

Concerning the time of motion.

4. The time of motion is obſerved by the beats of a pendulum which vibrates ſeconds: and the experiments, intended to illuſtrate the elementary propoſitions, may be eaſily ſo conſtructed that the time of motion ſhall be a whole number of ſeconds; the eſtimation of the time therefore admits of conſiderable exactneſs, provided the obſerver take care to let the bottom of the box A begin its deſcent preciſely at any beat of the pendulum; then the coincidence of the ſtroke of the box againſt the ſtage, and the beat of the pendulum at the end of the time of motion, will ſhew how nearly the experiment and the theory agree together. There might be various mechanical devices thought of for letting the weight A begin its deſcent at the inſtant of a beat of the pendulum; but the following method may perhaps be ſufficient: let the bottom of the box A, when at O on the ſcale, reſt on a flat rod held in the

hand horizontally, its extremity being coincident with o; by attending to the beats of the pendulum, and with a little practice the rod which ſupports the box A may be removed at the inſtant the pendulum beats, ſo that the deſcent of A ſhall commence at the ſame inſtant.

Of the velocity acquired.

5. It remains only to deſcribe in what manner the velocity acquired by the deſcending weight A, at any given point of the ſpace through which it has deſcended, is made evident to the ſenſes. The velocity of A's deſcent being continually accelerated will be the ſame in no two points of the ſpace deſcribed: this is occaſioned by the conſtant action of the moving force; and ſince the velocity of A at any inſtant, is meaſured by the ſpace which would be deſcribed by it, moving uniformly for a given time with the velocity it had acquired at that inſtant, this meaſure cannot be experimentally obtained, except by removing the force by which the deſcending bodies' acceleration was cauſed.

In order to ſhew in what manner this is effected practically, let us ſuppoſe that, according to a former example, the boxes A and B = 25 m each, ſo as together to be = 50 m; this with the wheel's in-

inertia 11 m will make 61 m: now let m be added to A, and an equal weight m to B, theſe bodies will balance each other, and the whole maſs will be 63 m. If a weight m be added to A motion will be communicated, the moving force being m, and the maſs moved 64 m. In a former ‖ example, the circular weight = *m was made uſe of as a moving force; but for the preſent purpoſe of ſhewing the velocity acquired, it will be convenient to uſe a flat rod, the weight of which is alſo = m‡. Let the bottom of the box A be placed on a level with o on the ſcale, the whole maſs being as deſcribed above = 63 m, perfectly balanced in equilibrio. Now let the rod, the weight of which = m, be placed on the upper ſurface of A; this body will deſcend along the ſcale preciſely in the ſame manner as when the moving force m was applied in the form of a circular weight. Suppoſe the maſs A to have deſcended by conſtant acceleration of the force m, for any given time or through a given ſpace: let a circular frame be ſo affixed to the ſcale, contiguous to which the weight deſcends, that A may paſs centrally through it, and that this circular frame may intercept the rod m by which

Fig. LXXX.

‖ Page 305.

* Fig. LXXX.

‡ Fig. LXXXII.

Fig. LXXXIII.

which the body A has been accelerated from quieſcence. After the moving force m has been intercepted at the end of the given ſpace or time, there will be no force operating on any part of the ſyſtem, which can either accelerate or retard its motion; this being the caſe, the weight A, the inſtant after m has been removed, muſt proceed uniformly ‡ with the velocity which it had acquired that inſtant: in the ſubſequent part of its deſcent, the velocity being uniform will be meaſured by the ſpace deſcribed in any convenient number of ſeconds, according to the examples given in the enſuing experiments.

‡ I. Law of motion.

Other uſes of the inſtrument.

It is needleſs to deſcribe particularly, but it may not be improper juſt to mention the further uſes of this inſtrument; ſuch as the experimental eſtimation of the velocities communicated by the impact of bodies elaſtic and nonelaſtic; the quantity of reſiſtance oppoſed by fluids, as well as for various other purpoſes; theſe uſes will be ſhewn on a future occaſion: but the properties of retarded motion being a part of the preſent ſubject, it may be neceſſary to ſhew in what manner the motion of bodies reſiſted by conſtant forces are reduced to experiment by means of the inſtrument above deſcribed, with as great eaſe and preciſion as the properties of bodies uniformly accelerated. A ſingle inſtance will be ſufficient: thus, ſuppoſe the maſs contained in the weights A and B and the wheels to be 61 m, when perfectly in equilibrio, as in a ‖ former example; let a circular weight m be applied to B, and let two long weights or rods, each = m, be applied to A, then will A deſcend by the action of the moving force m, the maſs moved being 64 m: ſuppoſe that when it has deſcribed any given ſpace by conſtant acceleration, the two rods m are intercepted by the circular frame above deſcribed, while A is deſcending through it; the ‡ velocity acquired by that deſcent is known, and when the two rods are intercepted, the weight A will begin to move on with the velocity acquired, being now retarded by the conſtant force m; and ſince the maſs moved is 62 m, it follows, that the force of * retardation will be $\frac{1}{62}$ part of that force whereby gravity retards bodies thrown perpendicularly upwards. The weight A will therefore proceed along the graduated ſcale in its deſcent with an uniformly retarded motion, and the ſpaces de-

Fig. LXXXIII.

‖ Page 305.

Fig. LXXX.

Fig. LXXXII.

‡ Sect. III. Prop. V.

* Sect. I. Prop. IX. p. 10.

deſcribed, times of motion, and velocities deſtroyed by the reſiſting force, will be ſubject to the ſame meaſures as in the examples of accelerated motion above deſcribed.

The uſes of the inſtrument, in verifying practically the properties of rotatory motion, will appear in the next ſection.

In the foregoing deſcriptions, two ſuppoſitions have been aſſumed, neither of which are mathematically true: but it may be eaſily ſhewn that they are ſo in a phyſical ſenſe; the errors occaſioned by them in practice being inſenſible:

1. The force which communicates motion to the ſyſtem has been aſſumed conſtant, which will be true only on a ſuppoſition that the line, at the extremities of which the weights A and B are affixed, is without weight. In order to make it evident, that the line's weight and inertia are of no ſenſible effect, let a caſe be referred to, wherein the body A deſcends through 48 inches from reſt by the action of the moving force m, when the maſs moved is 64 m: the time* wherein A deſcribes Fig. LXXVIII.

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* In theſe experiments the length of the line by which the weights are ſuſpended is about 72 inches, its weight

ſcribes 48 inches is increaſed by the effects of the line's weight by no more than $\frac{312}{10000}$ part of a ſecond; the time of deſcent being 3.9896 ſeconds, when the ſtring's weight is not conſidered, and the time when the ſtring's weight is taken into account = 4.0208 ſeconds; the difference between which is wholly inſenſible by obſervation.

2. The bodies have alſo been ſuppoſed to move in vacuo, whereas the air's reſiſtance

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3 grains: the vertical diſtance between the weights at the beginning of motion = 60 inches: ſuppoſe the ſpace deſcribed by the deſcending weight A = 48 inches, the maſs moved = $64\,m + 3$ grains = 7683 grains; wherefore in order to aſcertain the time in which the deſcending weight A deſcribes 48 inches from reſt, taking into account the weight and inertia of the line, we have by referring to prop. XIV. ſect. IV. $\mathcal{Q} = 7683$, $x = 48$ inches, $L = 72$, $b = 60$, $p = 3$ grains, w (or m) = 120, wherefore

$$\frac{\sqrt{px} + \sqrt{Lw + px - pb}}{\sqrt{Lw - bp}} = \frac{\sqrt{144} + \sqrt{8640 + 144 - 180}}{\sqrt{8640 - 180}}$$

$= \frac{104.75774}{91.97826} = 1.13891$, the hyperbolic logarithm of which = .13010; wherefore the time wherein the deſcending weight deſcribes 48 inches $= \sqrt{\frac{L\mathcal{Q}}{lp}} \times 13010 =$ $\sqrt{\frac{72 \times 7683}{193 \times 3}} \times .13010 = 4.0208$ ſeconds. If the ſtring's weight and inertia be not conſidered, the time will be =* $\sqrt{\frac{48 \times 7680}{120 \times 193}} = 3.9896$, and the difference between the times = 0.0312 parts of a ſecond.'

* Sect. III. Prop. IV. Cor. 5.

ſiſtance will have ſome effect in retarding their motion: but as the greateſt velocity communicated in theſe experiments, cannot much exceed that of about 26 inches in a ſecond, (ſuppoſe the limit 26.2845) and the cylindrical boxes being about $1\frac{3}{4}$ inches in diameter, the air's reſiſtance can never increaſe the time of deſcent in ſo great a proportion * as that of 240 : 241; its effects therefore will be inſenſible in experiment.

The

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* If the diameter of a cylinder moving in the direction of its axis be d, $p = 3.14159$,* z the altitude through which a body muſt fall freely by the acceleration of gravity to acquire the cylinder's velocity, the reſiſtance which the air oppoſes to the motion of the cylinder = the weight of $\frac{zpd^2}{4}$ cubic inches of air; that is, ſince according to the preſent caſe, $d = 1.75$, ‡ $z = \frac{\overline{26.2845}^2}{4 \times 193} = .8949$ parts of an inch, and $\frac{2}{7}$ of a grain, being equivalent in weight to each cubic inch of air, the reſiſtance to the two cylinder's motion $= \frac{.8949 \times 3.14159 \times \overline{1.75}^2 \times 2 \times 2}{4 \times 7} = 1.23$ parts of a grain. This is on a ſuppoſition, that the reſiſtance is always equal to that which oppoſes the weights, when moving at the rate of 26.2845 inches in a ſecond; and ſince the velocities are in every point of the deſcents leſs than that above mentioned, it is wholly needleſs to eſtimate the exact increaſe of time ariſing from the air's reſiſtance, which might if neceſſary be obtained from prop. XI. ſect. V. Suppoſing therefore the air's reſiſtance to be al-

* Sect. V. Prop. II. Cor. 2.

‡ Sect. III. Prop. V.

ways

The effects of friction are almoſt wholly removed by the friction wheels: for when the ſurfaces are well poliſhed and free from duſt, &c. if the weights A and B be balanced in perfect equilibrio, and the whole maſs conſiſts of 63 m, according to the example already ||deſcribed, a weight of $1\frac{1}{2}$ grains, or at moſt 2 grains, being added either to A or B, will communicate motion to the whole, which ſhews that the effects of friction will not be ſo great as a weight of $1\frac{1}{2}$ or 2 grains. In ſome caſes, however, eſpecially in experiments relating to retarded motion, the effects of friction become ſenſible; but may be very readily and exactly removed by adding a ſmall weight 1.5 or 2 grains to the deſcending body, taking care that the weight added is ſuch as is in the leaſt degree ſmaller than that which is juſt ſufficient to ſet the whole in motion, when A and B are equal and balance each other, before the moving force is applied. || Page 305.

In

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ways ſo great as 1.23 grains, and the moving force = 120 grains = m, the moving force will be diminiſhed by the air's reſiſtance in the ratio of 120 to 120 — 1.23, or of 120 to 118.77, and the time will be increaſed in a ſubduplicate proportion of 118.77 to 120, which differs not greatly from that of 240 to 241.

In the following deſcription of experiments, at leaſt of the more elementary ones, the ſpaces deſcribed, times of deſcription, &c. are ſo proportioned as to produce reſults in whole numbers, and are of ſuch a magnitude as admit of the moſt eaſy and convenient obſervation. The reſults are ſet down according to the theory from which the experiments will differ inſenſibly, if due care be taken in conſtructing and adjuſting the inſtrument.

N. B. The ſpace which bodies deſcribe by falling freely from reſt in 1 ſecond is = 193 inches; but in the enſuing experiments, this ſpace is aſſumed = 192 inches, (which will be productive of no ſenſible error) in order to avoid fractions, which would render the uſe of the inſtrument leſs eaſy and intelligible: in thoſe experiments at the end of the ſection, which are intended to illuſtrate propoſitions of a more complex nature, this aſſumption is uſeleſs; in theſe caſes therefore the ſpace through which bodies deſcend from reſt by their natural gravity in one ſecond is taken = 193 inches, its true value.

I.

If a quantity of matter = 64 m be acted on by a force = m, the ſpace deſcribed from reſt in one ſecond will be three inches.

If the quantity of matter denoted by 64 *m* be acted on by its natural gravity or weight 64 *m*, it will deſcribe from reſt 192 inches in one ſecond of time: and if the ſame quantity of matter 64 *m* be acted on by a force *m*, being 64 times ſmaller than before, it will || deſcribe only a ſixty-fourth

|| Sect. III. Prop. IV. Cor. 3.

fourth part of the former ſpace, that is, it will deſcribe only three inches in a ſecond from reſt, which may be experimentally illuſtrated by the inſtrument above mentioned. Suppoſing the boxes A and B to be each $= 6m$ in weight, let $19m$, or $4\frac{3}{4}$ oz. be included in each; then will A and B together $= 50m$: alſo, apply two circular weights $2m$ upon A, and one of theſe weights or m upon B, then will the maſs moved be $50m + 3m + 11m = 64m$, (the maſs $11m$ being the inertia of the ‡ wheels); and ſince $A = 27m$, and $B = 26m$, the moving force will be m. Having affixed the ſquare ſtage to 3 on the graduated ſcale, let the || under ſurface of the weight A begin to deſcend from 0 on the graduated ſcale at the inſtant of any beat of the pendulum: it will be heard to ſtrike the ſtage at the next ſucceſſive beat of the pendulum, having deſcribed three inches in one ſecond from reſt.

Fig. LXXVIII.

‡ Page 303.

|| Page 308.

Another example may be ſubjoined; every thing elſe remaining as in the laſt example, remove the circular weight m from B and place it on A, which will give $A = 28m$, $B = 25m$, then will the moving force $= 3m$, and the maſs moved $25m + 28m + 11m = 64m$, and the ſpace deſcribed from reſt by A in one ſecond $= \frac{3 \times 192}{64} = 9$ inches, which is experimentally verified in the ſame manner as was juſt deſcribed.

II.

Let the quantity of matter 64 m be put in motion by the action of the conſtant force m. *If the times of motion be 1 ſecond, 2 ſeconds, and 3 ſeconds, the ſpaces deſcribed from reſt by the deſcending weight A in thoſe times will be 3 inches, $3 \times 4 = 12$ inches, and $3 \times 9 = 27$ inches reſpectively; the ſpaces deſcribed from reſt being in a duplicate ratio of the times of motion.

* Sect. III. Prop. IV. Cor. 1.

Here let $A = 27\,m$, and $B = 26\,m$: this experiment is made exactly in the ſame manner as the laſt by ſetting the ſtage to 3, 9, and 27 on the ſcale, in the different experiments. Theſe practical examples ſhew, that if the accelerating forces be equal, the ſpaces deſcribed from reſt are in a duplicate ratio of the times of motion.

Fig. LXXVIII.

The following experiments alſo will illuſtrate this truth: let $36\frac{3}{4}\,m$ be included in A, and $36\frac{1}{4}\,m$ in B, then the experiments being made in the ſame manner as before, the ſpaces deſcribed in the times of deſcent 1 ſecond, 2 ſeconds, &c. will be as in the ſubjoined table.

Moving force.	Maſs moved.	Accelerating force.	Times of motion in ſeconds.	Spaces deſcribed from reſt eſtimated in inches.
$\frac{1}{2}\,m$	$96\,m$	$\frac{1}{192}$	1	1
			2	4
			3	9
			4	16
			5	25
			6	36
			7	49
			8	64

III.

If the quantity of ‡ matter 64 m be acted on by different moving forces, viz. m, 2 m, and 3 m, for any given time, for example 2 ſeconds, the ſpaces deſcribed

‡ Sect. III. Prop. IV. Cor. 3.

will be 12 inches, 24 inches, and 36 inches reſpectively, being in the ſame proportion with the moving forces.

Fig. LXXVIII. 1. Let $A = 27m$, $B = 26m$, then will the moving force $= m$, and the maſs moved $= 64m$: let the ſtage be placed at 12 on the graduated ſcale, and the under ſurface of *A* being coincident with 0, let it begin to deſcend at any beat of the pendulum; the weight will be heard to ſtrike the ſtage at the next beat but one, having deſcribed 12 inches in 2 ſeconds.

2. Let $A = 27\frac{1}{2}m$, and $B = 25\frac{1}{2}m$, then will the moving force $= 2m$, and the maſs moved $= 64m$; let the ſtage be ſet to 24 on the graduated ſcale, and by the ſame proceſs as was deſcribed in the laſt article, the weight *A* will be obſerved to deſcribe 24 inches from reſt in 2 ſeconds.

3. Let $A = 28m$, and $B = 25m$, then will the moving force $= 3m$, and the maſs moved $= 64m$, and the ſpace deſcribed from reſt by *A* in 2 ſeconds will be obſerved to $= 36$ inches.

It appears from theſe experiments, that when the times are the ſame, the ſpaces deſcribed from reſt are as the accelerating forces, or the quantities of matter being * given as the moving forces.

It being frequently neceſſary to vary the accelerating forces in the ſubſequent experiments, the following problem may be here inſerted, whereby the weights to be included in the boxes *A* and *B* may be in all caſes known, from having given the moving force and accelerating force required: ſuppoſe the given moving force were w, and the accelerating force were intended to be that part of the accelerating force of gravity which is expreſſed by the fraction $\frac{1}{n}$. Let x be the weight to be included in *B*, then will $x + w$ be the weight included in *A*: but the weight of the box $A = 6m =$ the weight of the box *B*, and the inertia of the friction wheels $= 11m$, wherefore the whole maſs moved will $= 2x + 23m + w$: and becauſe the given moving force is w, we have $\frac{w}{2x + 23m + w} = \frac{1}{n}$, and $x = \frac{nw - w - 23m}{2}$.

Thus,

* Sect. I. Prop. IX.

Thus, ſuppoſe it were required to aſſign what weight muſt be included in the boxes *A* and *B*, ſo that the force which accelerates the deſcent of *A* ſhall be $\frac{1}{96}$ part of the accelerating force of gravity, the given moving force being m; here referring to the rule $n = 96$, $w = m$, and $x = \frac{\overline{96-1-23}\times m}{2} = 36m$, or 9 oz. the weight included in *B*, and conſequently $37m$, the weight to be included in *A*.

IV.

If the quantity of matter 64 m be acted on by a moving force $= \frac{1}{2}$ m, it will deſcribe * from reſt 54 inches in 6 ſeconds; if the moving force be increaſed in the ratio of 1 : 4, ſo as to be = 2 m, the quantity of matter or 64 m not being altered, the time of deſcribing from reſt 54 inches, will be diminiſhed in the ratio of 2 : 1, ſo as to become only 3 ſeconds.

Let $20\frac{3}{4}m$ be included in *A*, and $20\frac{1}{4}m$ in *B*, then will the maſs moved $= 64m$, and the moving force $\frac{1}{2}m$, ſo that the force of acceleration $= \frac{1}{128}$, and the ſpace deſcribed from reſt in 6 ſeconds $= \frac{192\times 36}{128} = 54$ inches.

The latter part of the experiment ſhews, that if the ſpace deſcribed remains the ſame, while the time of deſcription is diminiſhed, the force of acceleration muſt be increaſed in a duplicate proportion of the times' diminution.

It has been ſufficiently deſcribed in what manner to diſpoſe the weights included in the boxes *A* and *B*, ſo that the accelerating force ſhall be of any aſſigned portion of the

* Sect. IV. Prop. IV. Cor. 5.

the accelerating force of gravity, and the methods of making theſe experiments having been illuſtrated by ſeveral examples, it may be ſufficient in the ſubſequent experiments juſt to mention the numbers ſuited to ſuch caſes, as may be moſt conveniently repeated : but theſe experiments may be varied at pleaſure.

V.

Let the quantity of matter 64 m, be acted on by a moving force* m for any given time, for example 2 ſeconds, the deſcending weight A will deſcribe 12 inches, and at the end of that deſcent will have acquired ſuch a velocity as, if uniformly continued, will cauſe the body to deſcribe 24 inches in the ſame time, i. e. in 2 ſeconds.

Fig. LXXXIII. Let $A = 26m$, and $B = 26m$, then will the weights A and B preciſely balance each other: in ſome former experiments, a circular weight m was applied to A as a moving force, and A was obſerved to deſcribe from reſt by conſtant acceleration ‡ 12 inches in 2 ſeconds: if inſtead of a circular weight m, applied as a moving force, a flat rod, the weight of which is alſo m, be placed on the upper ſurface of A, the acceleration of A's deſcent will be the ſame as before, i. e. ſuch as cauſes it to deſcribe 12 inches in 2 ſeconds from reſt. This being premiſed, let the circular frame, repreſented in fig. LXXXIII. be ſo affixed to the graduated ſcale, that the box A may in deſcending paſs centrally through it, and let its height be ſuch, that the inſtant the lower ſurface of the box A arrives at 12 on the graduated ſcale, the rod m may be intercepted by falling on the horizontal ſurface of the frame, and thereby be prevented from the further acceleration of the ſyſtem. Moreover, ſet the ſquare ſtage on the ſcale at 36 ; that is, 24 inches from the circular frame. The experiment will be as follows : let the weight A begin to deſcend from 0 on

Fig. LXXXII.

* Sect. III. Prop. II.

‡ Exp. II.

on the ſcale, at any beat of the pendulum. At the end of 2 ſeconds the rod will be heard to ſtrike againſt the circular frame, having deſcribed 12 inches with an uniformly accelerated motion, and at the fourth beat of the pendulum, the box *A* will ſtrike againſt the ſquare ſtage at 36, having deſcribed 24 inches in 2 ſeconds with an uniform motion.

Although it be certain, that the weight *A* deſcribed the latter 24 inches with a ‡ motion perfectly uniform, this may be further confirmed by ſetting the ſquare ſtage at 36, 48, and 60 ſucceſſively, every thing elſe remaining. Since the velocity is uniform, and *A* deſcribes 24 inches in 2 ſeconds, it will of courſe deſcribe 36 inches in 3 ſeconds, and 48 inches in 4 ſeconds: if the ſtage be fixed at 60 inches, and the experiment be repeated, the rod *m* will be removed at the ſecond beat of the pendulum as before, and the box *A* will ſtrike the ſtage at 60 at the ſixth beat of the pendnlum, having deſcribed 48 inches uniformly in 4 ſeconds.

‡ I. Law of motion. and p. 309.

In like manner it may be ſhewn experimentally, that during the motion of any body acted on by any conſtant force, a velocity is acquired, which if continued uniform will cauſe the body to deſcribe in the ſame time, a ſpace twice as great as that through which the body was accelerated, in order to acquire that velocity. Thus, let $A = 25\frac{1}{2} m$ and $B = 25\frac{1}{2} m$, then let two rods, each $= m$, be placed on the upper ſurface of *A*, and place the circular frame at 6, and the ſquare ſtage at 18; then the weight *A* beginning its deſcent at the inſtant of any beat of the pendulum, the two rods will ſtrike the frame, being removed from *A*, at the next ſucceſſive beat, and at the ſecond beat will ſtrike the ſtage underneath, having deſcribed 6 inches with an uniformly accelerated, and 12 inches with an uniform motion in equal times, i. e. 1 ſecond.

The method of affixing the circular frame, through which the weight *A* in its deſcent paſſes, ſhould be here deſcribed. Suppoſe, as in the preceding example, it were required to ſet the frame at the diviſion 12 on the graduated ſcale; it is meant to be ſo affixed, that the inſtant the rod *m*, deſcending along with the weight *A*, ſtrikes the circular frame, the lower ſurface of the box (by which the ſpaces deſcribed are always eſtimated) ſhall arrive at the diviſion 12. In order to effect which, when *A* and *B* are in equilibrio, and *A* is lower than the circular frame, as is repreſented in fig. LXXXIII. add any weight, for example *m* to *B*, then will *B* preponderate, and *A* will aſcend: if the rod *m* or other flat ſubſtance be held firmly over the circular aperture, it will prevent the weight *A* from aſcending further, its upper ſurface being now on a level with the horizontal ſurface of the circular frame: in this ſituation the weight *A* muſt be kept, until by altering the poſition of the frame, the under ſurface of the box *A* ſhall be on a level with the diviſion 12, or any other diviſion which the experiment may require: but it would be ſtill more eligible to have an index on that part of the frame which is contiguous to the diviſions, and at a diſtance from the frame's horizontal ſurface, equal to the length of the box *A*; by this means the circular frame may be truly adjuſted without further trouble, than ſetting the index to the proper diviſion on the ſcale.

VI.

If a quantity of matter = 64 m be act-
† Sect. III. Prop. I. ed on by any conſtant † force m, during the different times 1 ſecond, 2 ſeconds, 3 ſeconds, &c. the velocities generated will be thoſe of 6 inches, 12 inches, and 18 inches in a ſecond reſpectively, being in the ſame proportion with the times wherein the given force acts.

|| Exp. V. 1. The conſtruction of the preceding || experiment remaining, let the circular frame be ſet to 3, and the ſtage
Fig. LXXXIII. underneath at 9; the weight *A* beginning its deſcent at any beat of the pendulum, the rod will ſtrike the horizontal circular frame at the next beat, and the rod *m* being at that inſtant removed, the weight *A* will deſcribe the remaining 6 inches uniformly in 1 ſecond, ſtriking the ſtage at 9 at the ſecond beat.

2. Set the circular frame at 12, and the ſtage at 36, and the weight *A* will deſcend by conſtant acceleration through the firſt 12 inches in 2 ſeconds, and the remaining 24 inches in 2 ſeconds with an uniform velocity of 12 inches in a ſecond.

3. Set the circular frame at 27, and the ſtage at 45, and the weight *A* will deſcribe the 27 firſt inches from

reſt by conſtant acceleration in 3 ſeconds, and the remaining 18 inches in 1 ſecond, with an uniform velocity.

Theſe experiments illuſtrate the propoſition* wherein it is demonſtrated, that if the force by which bodies are accelerated is the ſame, the velocities generated are in the ſame proportion as the times wherein the given force acts.

VII.

If a quantity of matter = 64 m be acted on for any given ‡ time, for example 1 ſecond, by the different forces m, 2 m, and 3 m, the velocities generated will be 6 inches, 12 inches, and 18 inches in a ſecond, being in the ſame proportion with the forces which accelerate the maſs moved.

1. Let $A = 26m$ and $B = 26m$, and let the rod $= m$ be applied to A as a moving force; then the experiment being made as before deſcribed, the force m acting for 1 ſecond, generates in the maſs $64m$ a velocity of 6 inches in a ſecond. Fig. LXXXII. Fig. LXXXIII.

2. Let $A = 25\frac{1}{2}m$ and $B = 25\frac{1}{2}m$, and let two rods, each $= m$, be applied to A as a moving force; then will the maſs moved be $64m$: ſet the circular frame at 6, and the ſtage at 18; and the experiment being made as before, the two rods will be intercepted when the under ſurface of the weight A arrives at 6, and the remaining 12 inches of the deſcent will be deſcribed uniformly in 1 ſecond, which ſhews that the force $2m$ acting on the maſs $64m$ for 1 ſecond, generates in it a velocity of 12 inches in a ſecond.

3. Let $A = 25m$ and $B = 25m$, and let three rods, each equal to m, be applied to A as a moving force; then will the maſs moved be $64m$ as before: ſet the circular frame at 9, then will the three rods be intercepted when the weight A has deſcended 1 ſecond from reſt at 0, and A will then go on uniformly deſcribing 18 inches in the next

* Sect. III. Prop. I.

‡ Sect. III. Prop. I.

next ſecond: this ſhews that a force $3m$, acting on a maſs $64m$ for 1 ſecond, generates in it ſuch a velocity as cauſes it to deſcribe uniformly 18 inches in a ſecond.

It appears therefore, that if different forces accelerate the ſame body from quieſcence during a given time, the velocities generated will be in the ſame proportion with thoſe forces. Theſe experiments may be varied at pleaſure, to ſhew that the velocities generated in any different quantities of matter, the time in which the forces act being the ſame, will be as the accelerating forces.

VIII.

‡ Sect. III. Prop. I.

If bodies be acted on by ‡ accelerating forces, which are in the proportion of 3 : 4, and for times, which are as 1 : 2, the velocities acquired will be in the ratio of 1×3 to 2×4, or as 3 to 8.

† Sect. VII. Exp. VI. Art. 1.

1. A † force m acting on a maſs $64m$ for 1 ſecond, generates a velocity of 6 inches in a ſecond; here the accelerating force $= \frac{1}{64}$.

2. Now let $A = 19m$ and $B = 18m$, then will the moving force $= m$, and the maſs moved $19m + 18m + 11m = 48m$, wherefore the accelerating force is $\frac{1}{48}$. Set the circular frame at 16, and the ſtage at 48: the weight A in deſcending by the force of the rod m will deſcribe 16 inches in 2 ſeconds with an uniformly accelerated motion, and the remaining 32 inches until it ſtrikes the ſtage with an uniform motion of 16 inches in a ſecond. The velocities generated therefore in theſe two examples, are as 6 : 16, or as 3 : 8. In theſe experiments the accelerating forces are $\frac{1}{64}$ and $\frac{1}{48}$, or as 3 : 4. The times during which the bodies are accelerated from reſt 1 : 2: add theſe ratios the ſum will be the ratio of the velocities generated, or that of 3 : 8.

Fig. LXXXIII.

IX.

If a quantity of matter 64 m be accelerated through the ſame ‡ſpace, i. e. 12 ‡ Sect. III. Prop. V. inches by the forces m and 4 m, the velocities generated will be in a ſubduplicate ratio of theſe forces, or in the ratio of 1 to 2.

1. Let $A = 26m$ and $B = 26m$, and let a rod m be applied to A as a moving force, the maſs moved will be $64m$; then if the circular frame be ſet to 12, as in a former experiment, and the ſtage at 36; the weight A will deſcend through 12 inches in 2 ſeconds, and the rod m being then removed will proceed with an uniform velocity of 12 inches in a ſecond, to the ſtage underneath at 36.

2. Let $A = 24\frac{1}{2}m$ and $B = 24\frac{1}{2}m$, and let 4 rods, each equal to m, be applied as a moving force; then the quantity of matter moved will be denoted by $64m$; wherefore the accelerating force $= \frac{4m}{64m} = \frac{1}{16}$, being increaſed in the ratio of 1 to 4: every thing elſe remaining as in the laſt article, the weight A will deſcend through the firſt 12 inches from 0 with an uniformly accelerated motion in a ſecond, and will deſcribe the remaining 24 inches with an uniform velocity in 1 ſecond; the velocities generated in theſe two examples, are 12 inches and 24 inches in a ſecond, being in the ratio of 1 : 2; whereas the ratio of the accelerating forces is that of 1 : 4, the ſpace through which the bodies are accelerated being the ſame in both caſes.

X.

If different quantities of ‖ matter 64 m ‖ Sect. III. Prop. VI. and 48 m be impelled through the ſame ſpace, i. e. 12 inches, and acquire the ſame velocity, the moving forces muſt be in the ſame ratio as the quantities of matter moved, that is, as 4 : 3.

Experiments to illuſtrate this truth may be comprized in the ſubjoined table.

Moving forces.	Quantities of matter moved.	Accelerating forces.	Spaces deſcribed in inches.	Velocities acquired in inches in a ſecond.
m	$64m$	$\frac{1}{64}$	12	12
$\frac{3m}{2}$	$96m$	$\frac{1}{64}$	12	12
$\frac{3m}{4}$	$48m$	$\frac{1}{64}$	12	12

It is collected from the two laſt ſets of experiments, that the moving forces, which impel bodies through the ſame ſpace, are in the joint ratios of the quantities of matter moved and ſquares of the velocities generated.

XI.

* Sect. III. Prop. IV. Cor. 2.

If a quantity of *matter 64 m be impelled through different ſpaces from reſt, 3 inches and 27 inches, by the action of the ſame force m, the velocities generated will be in a ſubduplicate ratio of the ſpaces deſcribed, or as 1 : 3.

1. If the maſs moved is 64*m* and the moving force *m*, the velocity generated in deſcending through 3 inches, has been ſhewn in a former experiment to be 6 inches in a ſecond.

2. Now let *A* deſcend by the ſame force through 27 inches, and ſet the circular frame to 27, and the ſtage at 45, the weight *A* will deſcribe the firſt 27 inches with an uniformly accelerated motion in 3 ſeconds, and the rod *m* being there removed will ſtrike the ſtage at 45 at the fourth beat of the pendulum, having deſcribed the laſt 18 inches uniformly in a ſecond. In theſe experiments the ſpaces deſcribed during the body's acceleration are 3 and 27, or as 1 : 9, the velocities acquired as 6 : 18, or as 1 : 3, being in a ſubduplicate ratio of the ſpaces.

It appears therefore that when the forces of acceleration are the ſame, the velocities acquired are in a ſubduplicate ratio of the ſpaces; and from a former ſet of * experiments it is collected, that if the ſpaces deſcribed are the ſame, the velocities generated are in a ſubduplicate ratio of the forces of acceleration: it may be concluded therefore, that when both ſpaces deſcribed and accelerating forces vary, the velocities ‡generated will be in a ſubduplicate ratio of the accelerating forces and of ſpaces deſcribed jointly.

* Exp. IX.

‡ Sect. III. Prop. V.

The laws obſerved during the motion of uniformly accelerated bodies are made evident to the ſenſes, by the experiments which have been already deſcribed: thoſe which follow are deſigned to illuſtrate the properties of uniformly retarded motion. When a body is thrown perpendicularly upward from the earth's ſurface, it is continually reſiſted by a force which is equal to the body's weight; and the weights of all ſubſtances being proportional to the quantities of matter contained in them, it follows, that the force which retards the perpendicular aſcent of any body being † meaſured by its weight divided by the aſcending maſs is the ſame, being ſuch as deſtroys a velocity of $32\frac{1}{6}$ feet in a ſecond, in each ſecond of the body's motion. But in order to illuſtrate by experiment the general laws according to which bodies are retarded by the action of conſtant forces, ſuch methods ſhould be made uſe of as will enable us to apply different reſiſting forces to the ſame maſs of matter and the ſame reſiſting force to different quantities of matter; both of which conditions will be ſatisfied by the inſtrument which has been already deſcribed, and will be again referred to in the ſubſequent experiments. In order to render theſe more obvious, a

† Sect. I. Prop. IX.

few conſiderations concerning reſiſtances may be inſerted previous to the deſcription of the experiments. A reſiſting force is to be underſtood to convey preciſely the ſame idea as the term moving force, except as far as regards the directions in which theſe forces act in reſpect of the bodies' motion, theſe directions being contrary to each other. To exemplify this, let a muſket ball of $\frac{3}{4}$ of an inch in diameter, and 1.3 oz. weight, deſcend toward the earth's ſurface by its own gravity: the moving force is in this caſe equal a weight of 1.3 oz. Let the ſame ball be thrown perpendicularly upward; the reſiſting force is likewiſe a weight of 1.3 oz. but let the ſame ball impinge perpendicularly againſt an immoveable block of elm, ſo as to penetrate its ſubſtance, the ball's motion will be reſiſted by a force equal to * 107000 × 1.3 oz. if therefore the earth's gravity were 107000 times greater than it really is, and the leaden ball before referred to were projected perpendicularly upward with any given velocity, the reſiſtance oppoſed to the ball's motion would be exactly the ſame, as when the ball is projected againſt an immoveable block of elm with the ſame velocity, and conſequently the ball ſo projected would riſe to a perpendicular altitude equal to the depth to which it would penetrate into the ſubſtance of the elm, if it impinged againſt the elm with the ſame velocity of projection.

Theſe principles admit of very exact illuſtration from experiments, which alſo confirm thoſe mechanical propoſitions, which † Leitnitz and Bernouilli, &c. deduced from obſerving the depths to which ſpherical bodies penetrated the ſubſtance of clays and other ſoft bodies. Theſe enſuing experiments will alſo be the more ſatisfactory from their admitting of an exact eſtimation of the moving force and the maſs moved, in which the experiments before mentioned are defective; neither is the force of reſiſtance conſtant in the experiments on the depths to which ſpheres penetrate into the ſubſtance of clays, on account of their ſpherical figure, and the law according to which the forces vary, it would be difficult to aſſign.

In the ſubſequent experiments, the force of reſiſtance is a weight, and therefore of invariable quantity, during the whole time it is oppoſed to a body's motion, and the the exact quantity of it is moreover eaſily aſſigned.

* p. 39.

† p. 333.

XII.

Sect. III.
Prop. VIII.

Let a given maſs 61 m be projected with a velocity of 18 inches in a ſecond: then if it be reſiſted by a conſtant force m, the ſpace deſcribed by the reſiſted maſs, before its motion is deſtroyed, will be 25.6; if it be reſiſted by a force = 2 m, the ſpace deſcribed will be $12.8 = \frac{25.6}{2}$, and if the reſiſtance be 3 m, the ſpace will $8.533 = \frac{25.6}{3}$ inches; theſe ſpaces being in the inverſe ratio of the reſiſting forces.

Let $A = 24\frac{1}{2}\,m$ and $B = 25\frac{1}{2}\,m$, and apply to the upper ſurface of A two rods, each of which $= m$, then will the weight A preponderate and deſcend by the action of a moving force $= m$, the whole maſs moved being $= 63\,m$. Set the circular frame to 26.44; then the weight A by deſcribing from reſt the ſpace 26.44 inches will acquire a velocity $= \sqrt{\frac{4 \times 193 \times 26.44}{63}} = 18$ * inches in a ſecond: and at that inſtant the two rods, each of which $= m$, being intercepted, the weight A will continue to deſcend with an uniformly retarded motion; which will be preciſely the ſame as if a maſs of $61\,m$, without gravity, were projected with a velocity of 18 inches in a ſecond in free ſpace, and a force of reſiſtance $= m$ were oppoſed to its motion; wherefore A (with the other parts of the ſyſtem) will loſe its motion gradually, and will deſcribe a ſpace $= \frac{18^2 \times 61}{4 \times 193} = 25.6$ inches before its motion is entirely deſtroyed: A will therefore be obſerved in the experiment

* Sect. III. Prop. V. and p. 32.

to deſcend as low as 52 inches, before it begins to aſcend by the ſuperior weight of *B*.

2. Let $A = 24m$ and $B = 26m$, and let three rods, each equal to *m*, be applied at the upper ſurface of *A*, and ſet the circular frame at 26.86; then will *A* deſcend by a moving force *m*, the whole maſs moved being $64m$, and conſequently while *A* deſcends through 26.86 inches, it will acquire a velocity of 18 inches in a ſecond: and the three rods being intercepted by the circular frame, *A* will go on with an uniformly retarded motion, being reſiſted by a force $= 2m$, and the maſs moved $= 61m$, wherefore the force of retardation being $= \frac{2}{61}$, the ſpace deſcribed by *A* before its motion is ‡ deſtroyed $= \frac{18^2 \times 61}{4 \times 193 \times 2} = 12.8$ inches; and will therefore be obſerved from the experiment to deſcend as low as the diviſion on the ſcale which correſponds to 39.66 inches.

3. Let $A = 23\frac{1}{2}m$ and $B = 26\frac{1}{2}m$, and let four rods, each of which $= m$, be applied on the upper ſurface of *A*, and ſet the circular frame to 27.28; then will *A* deſcend by the force *m*, the whole maſs moved being $65m$, and the four rods will be intercepted by the circular frame at 27.28, and *A* having acquired at that inſtant the velocity of 18 inches in a ſecond, will begin to be reſiſted by a force of $3m$, the maſs moved being $61m$, and *A* will deſcribe a ſpace $= \frac{18^2 \times 61}{4 \times 193 \times 3} = 8.533$ inches, before its motion is deſtroyed: *A* will therefore be obſerved in the experiment to deſcend to the diviſion on the ſcale which anſwers to 35.81.

In the ſame manner, other principles relating to retarded motions may be verified experimentally. It is well known, that if ſpheres of the ſame magnitude, but of different ſpecific gravities fall on an horizontal ſurface of clay or any ſimilar ſubſtance, having deſcended from altitudes which are in an inverſe proportion of their ſpecific gravities, that the depth to which they penetrate will be equal. To analyze this is not difficult; for aſſuming the force which reſiſts the progreſs of the balls as conſtant, it will be of the ſame quantity when oppoſed to the different balls, becauſe their magnitudes † are equal; and ſince the quantities of matter contained in the balls are directly as their ſpecific gravities, the force which retards the balls will be inverſely as their ſpecific gravities: let therefore $\frac{F}{f}$ repreſent the direct ratio of the forces which retard any two of the ſpheres, or the inverſe ratio of their ſpecific gravities: $\frac{A}{a}$ the ratio of the altitudes from which they fall, $\frac{V}{v}$ the ratio of the velocities with which they impinge on the ſurface of the ſubſtance which they penetrate to the depths of S, s reſpectively; then from the principles of uniformly retarded motion, we have *$\frac{F}{f} \times \frac{S}{s} = \frac{V^2}{v^2}$‡$= \frac{A}{a}$, which is $= \frac{F}{f}$ by the conſtruction of the experiment; wherefore $\frac{S}{s} = 1$, or the ratio of the depths to which they penetrate the ſubſtance on which they impinge will be that of equality. Whenever therefore the retarding forces are in a direct duplicate ratio of the velocities wherewith bodies begin to move, the ſpaces deſcribed by them with an uniformly retarded motion will be equal. Thus, let a maſs $= 64m$ be projected with a velocity of 12 inches in a ſecond, and let it be reſiſted by a force m; moreover, let a maſs $48m$ be projected with a velocity of 24 inches in a ſecond, and ſuppoſe it to be reſiſted by a force $= 3m$, then will the force which retards the motion of the bodies be as $\frac{1}{64}$ to $\frac{3}{48}$, or as 1 : 4: and becauſe the velocities of projection are as 1 : 2, the ſpaces deſcribed by the uniformly retarded bodies will be equal, being in each caſe 11.94 inches.

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‡ Sect. III. Prop. V. and p. 32.

† p. 40.

Theſe experiments are eaſily made, as in the preceding caſes, and the adjuſtment of the weights neceſſary may ſerve as an example to render the uſe of the inſtrument familiar.

In theſe inſtances the motion of the retarded body is continued, until it is gradually deſtroyed by the reſiſting force: it may be proper to ſubjoin one practical example concerning retarded motion, wherein the body A, moving with a known velocity, is retarded by a given force, and the time of deſcribing any given ſpace eſtimated from the beginning of its retarded motion is obſerved experimentally.

* Sect. III. Prop. V.

‡ Sect. III. Prop. IV. Cor. 2.

XIII.

Let a maſs = $63\frac{1}{2}$m move with a velocity of 11.87 inches in a ſecond; if it be retarded by a reſiſting force $\frac{1}{2}$ m, it will deſcribe 21.95 inches in 3 ſeconds, the beginning of motion being eſtimated from the firſt inſtant of its retardation.

Let $A = 26m$ and $B = 26\frac{1}{2}m$, and let a flat rod, ſimilar to thoſe which have been before referred to (but the weight muſt in this inſtance be $1\frac{1}{2}m$) be applied on the upper ſurface of A, and ſet the circular frame to 11.877 on the ſcale; then will A deſcend, the moving force being m and the maſs moved $= 65m$, and when it has deſcribed 11.877 inches, it will have acquired a velocity of 11.877 inches in a ſecond; and the rod $1\frac{1}{2}m$ being at that inſtant intercepted, the maſs A will begin to deſcend with an uniformly retarded motion, the reſiſting force being $\frac{1}{2}m$, and the maſs moved $63\frac{1}{2}m$, and conſequently the force of retardation $\frac{.5}{63.5} = \frac{1}{127}$; and ſince if the retarding force $= F$, and the initial velocity be equal to that which a body acquires in falling from reſt freely by its natural gravity through the ſpace b, and $l = 193$ inches, the ſpace deſcribed in T ſeconds $= 2T \times \sqrt{bl} - lFT^2$: this * rule being applyed to the preſent caſe, gives $T = 3$, $b = .1827$ part of an inch, being the ſpace through which a body deſcends freely from reſt, by the acceleration of gravity, to acquire the velocity of 11.877 inches in a ſecond, $F = \frac{1}{127}$, and the ſpace required $= 6 \times \sqrt{.1827 \times 193} - \frac{9 \times 193}{127} = 21.95$ inches.

* Sect. III. Prop. XI.

If therefore the ſquare ſtage be affixed to 33.83, and the weight *A* begins to deſcend from 0, at any beat of the pendulum, the rod $1\frac{1}{2}\,m$ will ſtrike the circular frame at the end of two ſeconds, having deſcribed 11.877 with an uniformly accelerated motion, and the rod being intercepted, *A* will go on with an uniformly retarded motion, and will ſtrike the ſtage at 33.83 at the fifth beat, having deſcribed the laſt 21.95 inches in 3 ſeconds.

The following articles may tend to facilitate the conſtruction and adjuſtment of the inſtrument repreſented in Fig. 78 and 83 and referred to throughout this ſection.

1. The pendulum of the clock which is affixed to the pillar of the inſtruments vibrates ſeconds; it has only one wheel which immediately acts on the pendulum: the ſmall weight which continues the pendulum's motion, after it has been wound up, is half an hour in deſcending to the ground. The clock will be ſufficiently exact if it keeps time with a common well regulated clock for this half hour.

2. The feet of the frame on which the friction wheels reſt, ſhould not be fixed immoveably to the frame underneath, becauſe in order to adjuſt the axis of the wheel, *a b c d* horizontal, it will be neceſſary to alter the height of the two anterior or poſterior feet, by inſerting ſmall plates of wood or braſs under them, or by means of ſcrews acting immediately on the feet. — A hanging level applied to the wheel's axle, will ſhew when the axis is horizontal.

3. When the axis of the wheel *a b c d* has been adjuſted horizontal, let two equal weights *A* and *B* be ſuſpended from the extremities of a ſilk line of proper length, the thickneſs of which is no greater than is juſt ſufficient to ſuſtain the weights. When theſe weights are perfectly quieſcent, a ſmall impulſe being applied to either, in a vertical direction, will ſet the whole in motion; which will be continued uniform till one of the boxes arrives at the extremity of the ſcale. When the box *A* is at the bottom of the ſcale and quieſcent, it muſt be obſerved whether the middle line on the ſcale is every where exactly oppoſite to the line ſuſtaining *A*; or in other words, whether the line in the middle of the ſcale is in the ſame vertical plane with the line which ſuſtains *A*. If it is not, the lower extremity of the ſcale muſt be moved along the arm of the baſe until the adjuſtment is correct. 2. It is alſo to be obſerved whether the line is every where at equal perpendicular diſtances from the Fig. LXXVIII.

the middle line on the ſcale: if it is not, the upper extremity of the ſcale, muſt be moved further from or nearer to the ſilk line until the diſtances are every where equal. Fig. LXXXIII. The middle line on the ſcale will now be vertical, and the circular frame deſcribed in page 310, muſt be ſo conſtructed, that the box *A* may paſs centrally through it when theſe adjuſtments are correct.

4. The wheel *a b c d* previous to its adjuſtment, ſhould weigh ſomething more than ſix ounces, provided the figure of the wheels be not very different from thoſe repreſented in the Fig. 78, 83. In the circumference of the wheel *a b c d*, a ſmall groove ſhould be cut and well poliſhed, to admit the line which goes over the wheel and ſuſtains the weights *A* and *B*.

5. The weight of the wheel *a b c d* muſt be ſo adjuſted that the reſiſtance to motion of its inertia, with that of the friction wheels ſhall be equivalent to $2\frac{3}{4}$ ounces. This may be effected according to the method deſcribed in page 301, that is, by winding a very ſlender line round the wheel, and applying to the extremity a weight of 30 grains: let the ſquare ſtage be fixed to the ſcale at $38\frac{1}{2}$ inches, and let the weight 30 grains, begin to deſcend at any beat of the pendulum from 0 on the ſcale. If it deſcribes the $38\frac{1}{2}$ inches in leſs than three ſeconds, the wheel is too light to admit of the experiments being made according to the numbers uſed in this ſection. Suppoſe then the weight to be more than 3 ſeconds in deſcribing the $38\frac{1}{2}$ inches from reſt: the wheel's weight muſt be gradually diminiſhed until the time of deſcent is exactly three ſeconds. It is however to be noted that in firſt making this experiment for the adjuſtment of the wheel's weight, the line was wound round the cylindrical ſurface of the wheel's circumference, no groove having been at firſt found neceſſary. When a groove is cut the line muſt no longer be wound round the wheel for reaſons too obvious to need mentioning: in this caſe therefore the adjuſtment of the wheel's weight is to be obtained by another method. Inſtead of the weights *A* and *B*, let the weights 48 grains and 24 grains be ſuſpended from the extremities of the line which goes over the wheel *a b c d*, and ſet the ſquare ſtage at 30; then letting the weight 48 grains, begin to deſcend from 0 on the ſcale at any beat of the pendulum, obſerve the time of its deſcribing the 30 inches as meaſured on the ſcale: if the time of deſcent ſhould be leſs than three ſeconds the wheel is too light; ſuppoſe therefore the time to be greater than three ſeconds; the wheel's

weight muſt be gradually diminiſhed until the time of deſcent is preciſely three ſeconds. When this adjuſtment is correct, the reſiſtance from the wheel's inertia including that of the fraction wheels, will be exactly the ſame as if thoſe wheels were removed and the maſs 2¾ oz, was uniformly accumulated in the circumference of the wheel to which the line is applied.

6. Great attention ſhould be paid to the adjuſtment of the weights uſed in theſe experiments as a very ſmall error ſuch as is ſcarcely perceivable in each, will tend greatly to affect the exactneſs of the experiments in which many weights are uſed.

7. In letting the box *A* begin to deſcend at any beat of the pendulum according to the method deſcribed in page 308, the obſerver muſt not wait until he hears the the beat, at which he intends *A*'s deſcent ſhall begin; for in this caſe *A*'s deſcent will always commence too late: the proper method is to attend to the beats of the pendulum until an exact idea of their ſucceſſion is obtained; then the extremity of the rod being withdrawn from the bottom of the box *A* directly downward at the inſtant of any beat, the deſcent will commence at the ſame inſtant.

This inſtrument was executed with great mechanical ſkill, partly by Mr. *L. Martin*, and partly Mr. *G. Adams*, mathematical inſtrument makers in London.

SECT. VIII.

CONTAINING A DESCRIPTION OF EXPERIMENTS ON THE MOTION OF BODIES WHICH REVOLVE ROUND FIXED AXES, AND ROUND SUCH AS MOVE WITH THE SAME VELOCITIES AS THE CENTRES OF GRAVITY OF THE REVOLVING BODIES.

ALTHOUGH experiments may be made on bodies or ſyſtems of bodies, which revolve round fixed axes inclined to the horizon at any different angles, it is however moſt convenient that the axes ſhould be either horizontal or vertical.

Which of theſe poſitions ſhould be preferred to the other, will depend on the circumſtances of the different experiments; chiefly on the weights and figures of the parts which compoſe the revolving ſyſtem. When the axle is horizontal it is abſolutely neceſſary that it ſhould be ſupported on friction wheels, which greatly diminiſh, or altogether prevent the loſs of motion which would be cauſed by the friction of the axle, if it revolved on an immoveable ſurface: If therefore the the weight of the ſyſtem be ſuch as can be ſupported eaſily on friction wheels, and the parts of it be of proper ſhape and dimenſions, the poſition of the axle may be horizontal. In Fig. 75. P L Q repreſents a wheel and axle which in the experiments reſt on the friction wheels*. Suppoſe that two weights B and A act on the wheel and axle reſpectively, and that B preponderates againſt A, deſcending in a vertical line; the graduated ſcale muſt be adjuſted ſo that the ſpace deſcribed by B in its deſcent may be meaſured on it, or if A preponderates againſt B, the ſcale muſt be placed ſo as to become a meaſure of A's deſcent: by this means the ſpaces deſcribed, times of deſcription, real velocities of the deſcending weight as well as the angular velocities generated in the ſyſtem, will be experimentally obtained. In other caſes the ſyſtem will revolve on a vertical axis as in Fig. 84. where A B repreſents a cylindrical axle; C D a cylindrical rod, the axis of which cuts the vertical axis at right angles; C O, D Q are two equal ſpheres, through the centres of which the axis of the cylinder paſſes; theſe ſpheres are faſtened to the cylinder by ſcrews, at equal diſtances from the vertical axis: motion is communicated to the ſyſtem by the line

Fig. LXXVIII.

E F, ſtretched by a weight p; the line E F having been wound round the vertical axle paſſes horizontally and goes over a wheel or Fig. LXXVIII. pully F G,* which revolves on the friction wheels ſo often referred to in the laſt ſection. It is eaſy to ſee that the frame U X Y Z muſt be faſtened to the continuation of the ſtage on which the friction wheels are placed, i. e. to that part of the ſtage which projects beyond the pillar.

Fig. LXXVIII. The graduated† ſcale is to be ſo adjuſted as to meaſure the ſpace deſcribed by p, as it deſcends by communicating motion to the revolving ſyſtem. The repreſentation of the frame U X Y Z, and the ſyſtem C D A B is omitted in Fig. 78, and 81. for want of room.

In the enſuing experiments, the moving force is applied at the circumference of a wheel or axle, the axis of which coincides with the axis of the ſyſtem; it is therefore neceſſary that the diſtance of this circumference from the axis, or the radius of the axle, ſhould be very preciſely known; becauſe the quantity of the moving force, exerted by a given weight depends entirely on it. In direct menſuration, errors muſt be unavoidably committed which bear a conſiderable proportion to the whole, eſpecially when the radius to be meaſured is ſmall; in order to obviate this difficulty, the following indirect method was uſed, by which the radii of the cylinders uſed in the experiments were obtained to very great exactneſs.

Having fixed to the extremity of a very fine and flexible line a weight ſufficient to keep the line ſtretched, faſten the other extremity to the axle of which the radius is required; the line being ſtretched by the weight mentioned above, meaſure by a ſcale of even parts, any convenient length, 6 inches for example, and mark the extremities of the length ſo meaſured; then holding the axle horizontal, let the meaſured part of the line be wound round it in the form of an helix, the circumferences being every where contiguous. Count the number of revolutions, and ſuppoſe them $= n$, alſo meaſure the length of the cylinder occupied by the helix; let this $=$ a, and the length of the helix or line firſt mentioned $= l$, $p = 3.14159$, &c. then, the radius of the cylinder $= \dfrac{\sqrt{l_2 - a^2}}{2pn}$.

The exactneſs of this method may be known by obſerving, that if the cylinder be truly made, and the proceſs carefully repeated with different values of *l n* and *a*, the radius deduced will however always come out the ſame to the 4th or even the 5th decimal place.

I.

Fig. LXXV. In the wheel and axle P L Q, the radius of the wheel S L is to the radius of the axle S O, as 10 to 1: the diſtance of the centre of gyration from the axis S R = 6.7345 parts, the radius of the axle being eſtimated as one of them; the wheels' weight = 6.0635 oz. then if two weights A and B act againſt each other on the axle and wheel reſpectively, whereof A = 10 oz. and B = 1 oz. theſe weights will exactly balance each other. Let 48 grains or $\frac{1}{10}$ oz. be added to B, ſo that B ſhall become = 1.1 oz. it will deſcend, and will deſcribe from reſt 19.54 inches in two ſeconds.

Fig. LXXVIII. The wheel and axle *P L Q* is exactly equal in weight and dimenſions to the wheel and * axle *a b c d*, except only in having no groove cut in the circumference, which in this caſe is a plane cylindrical ſurface, terminated on each ſide by an edge, to prevent the line from falling off: this wheel and axle is to be placed on the friction wheels, and the weights being applied as deſcribed above, adjuſt the graduated ſcale ſo that it ſhall be oppoſite to the deſcending weight *B*, and ſet the ſquare ſtage at 19.54: then let *B* begin to deſcend from 0 on the ſcale at any beat of the pendulum, it will ſtrike the ſtage at the ſecond beat, having deſcribed 19.54 in two ſeconds.

In the deſcription of this experiment it is ſaid, that the diſtance of the centre of gyration from the axis is 6.7345 parts, the radius of the axle being one of them. This is obtained from having the equivalent weight, which being uniformly accumulated, in the circumference reſiſts the communication of motion by its inertia, equally with the maſs contained in the wheels: *This equivalent weight was found to be 2.75 oz. Suppoſe then the maſs of the wheels to be removed, and the 2.75 oz. to be accumulated uniformly in the circumference, at the diſtance from the axis $S\,L$, and let $2.75 = P$, and the wheels' weight W; then according to the rule demonſtrated in Prop. VIII. Sect. VI. $SR = \sqrt{\frac{P \times SL^2}{W}} = \sqrt{\frac{2.75 \times 100}{6.0635}} =$ 6.7345.

* p. 302.

To compare this experiment with the theory we muſt refer to Sect. VI. Prop. XIX. Cor. 1. which being applied to the preſent caſe, gives $n = 10$, $p = 1.1$, $q = 10$, $w = 6.0635$, $r = 6.7345$, and the force which accelerates the deſcent of $B = \frac{n^2 p - n q}{w r^2 + p n^2 + q} = \frac{110 - 100}{275 + 110 + 10} = \frac{1}{39.5}$. Wherefore the ſpace deſcribed from reſt by the deſcending weight B in two †ſeconds $= \frac{193 \times 4}{39.5} =$ 19.54.

† Sect. III. Prop. IV. Cor. 5.

This experiment may be varied by making B deſcend for 1 ſecond, 2 ſeconds, and 3 ſeconds: the ſpaces deſcribed from reſt, will be 4.88, 4.88 × 4 and 4.88 × 9, whereby it is practically ſhewn, that the ſpaces deſcribed are in a duplicate ratio of the times of motion; and conſequently that the rotation of the ſyſtem is equably accelerated.

II.

On the ſame wheel and axle P Q S L, let a weight A = 10 oz. be ſuſpended from the axle by means of a line wound round it, and B = 1 oz. from a line going round the

Fig. LXXV.

the circumference of the wheel ſo as to act againſt the former, theſe will balance each other. Now let $\frac{3}{4}$ oz. be added to A, ſo that it ſhall become = 10$\frac{3}{4}$ oz. then will A deſcend and will deſcribe 3.37 inches in 3 ſeconds, from reſt.

If the weight which deſcends from the axle be q, the weight acting againſt it on the circumference of the wheel $= p$, the radius of the wheel $= n$, that of the axle $= 1$, the diſtance of the centre of gyration from the axis $= r$, and the wheels weight $= w$; the force which accelerates the deſcent of $q = \frac{q - pn}{wr^2 + pn^2 + q}$ which is demonſtrated exactly in the ſame manner as Prop. XIX. Sect. VI. Applying this to the preſent caſe, we have $A = q = 10.75$, $B = p = 1$, $SL = n = 10$, $w = 6.0635$, $r = 6.7345$, and the force of acceleration $= \frac{10.75 - 10}{275 + 100 + 10.75} = \frac{.75}{385.75}$; wherefore the ſpace deſcribed by the deſcending weight A in one ſecond $= \frac{.75 \times 193}{385.75} = .3752$ parts of an inch, and in 3 ſeconds the ſpace will be $= .3752 \times 9 = 3.37$ inches.

In all the preceding experiments, the weights have been eſtimated according to the troy meaſure, the ounce conſiſting of 480 grains; in the enſuing experiments the avordupoiſe ounce happening to have been uſed when they were originally conſtructed, it was afterwards thought needleſs to alter the numbers; this is here mentioned, not becauſe it is of any conſequence to what ſtandard the weights in any experiments are referred as far as regards the weights themſelves, but becauſe according to the known ſpecific gravities of the ſubſtances uſed, the dimenſions and weights of the parts, of which the revolving ſyſtems conſiſt would not be conſiſtent with the numbers ſet down in the experiments, according to the troy meaſure; the ſtandard weight therefore referred to in the remaining experiments will be the averdupoiſe oz. conſiſting of 437.5 grains.

III.

A B O Q is the ſyſtem deſcribed in page 239, of which the dimenſions, &c. are as follows: Radius of the ſteel axle A B = .20133 parts of an inch, its weight = 2.4713 oz. the radius of the ſteel rod O Q = .0671, length of the rod O Q = 12.55; its weight = .8034 oz: radius of either braſs ſphere C O, D Q = .74, the weight of each ſphere = 7.985 oz. a line is wound round the axle A B, and paſſing horizontally over the wheel F G, is ſtretched by a weight p = 10 oz. which deſcribes from reſt during its deſcent 15.5 inches in ten ſeconds, or 5.59 inches in ſix ſeconds. Fig. LXXXIV.

The wheel *F G* as in the laſt experiment revolves on the friction wheels, and the ſpaces deſcribed by *p* in its deſcent are eſtimated by means of the graduated ſcale as before. Fig. LXXVIII. To compare this experiment with the theory, we may have recourſe to Prop. I. Cor. 2. Sect. VI. where it is ſhewn that in any revolving ſyſtem, the force which accelerates the point to which the moving force is applied, is that part of the acceleration of gravity which is expreſſed by a fraction, of which the numerator is the ſquare of the diſtance at which the force is applied from the axis, multiplied into the moving force, and the denominator the ſum of all the products formed by multiplying each particle of the ſyſtem into the ſquare of its diſtance from the axis of motion.

In the preſent caſe therefore it is neceſſary to obtain the ſum of all the products which can be formed by multiplying each particle of the ſyſtem *A B O Q* into the ſquare of its diſtance from the axis of motion, according to the dimenſions given.

In order to eſtimate the ſum of theſe products in the two ſpheres, ſome few conſiderations are neceſſary before the uſual rules can be applied. It is to be obſerved that in each ſphere there is a cylindrical cavity terminated at each end by portions of the ſphere; but theſe portions are far too ſmall to have ſenſible effect in the computation, even if it were carried beyond the uſual number of decimals. The cavities therefore being cylinders of which the length is equal to the ſphere's diameter or $= 1.48$, and their radius $= 0.0671$, and the ſpecific gravity of braſs being 8 when that of water is 1; it appears that if theſe cylinders were of braſs, the weight of each would be $=$ 0.0969 part of an oz. which being added to 7.985 the weight of each ſphere when the cavity is not filled up, will give the weight of the entire ſphere of braſs $=$ 8.0819 oz.

When any ſphere revolves round an axis of motion, from which the diſtance of the ſphere's centre is d, the radius of the ſphere being r and its weight w, the ſum of the products which can be formed by multiplying each † Page 224. particle into the ſquare of its diſtance from the axis of † motion $= \frac{5\,d^2 w + 2\,r^2 w}{5}$. Applying this to the preſent caſe, we have $d = SE = 5.535$, $r = .74$, $w = 8.0819$, and the ſum of the products required $=$

$$\frac{5 \times \overline{5.535}^2 \times 8.0819 + 2 \times \overline{.74}^2 \times 8.0819}{5} = 249.37;$$

but from this ſum of products juſt found muſt be ſubtracted the ſum of thoſe which are formed by multiplying each particle of the braſs cylinder, imagined to be inſerted in the ſphere, into the ſquare of its diſtance from the axis. ‡ Page 223. Now when any cylinder revolves round ‡ an axis of motion, the axis of the cylinder being perpendicular to that axis, the ſum of the products mentioned above may be obtained by the following rule. Let the radius of the cylinder $= r$, its weight w, the diſtance of its centre of gravity from the axis of motion $= d$; the diſtance of the neareſt extremity from the axis of motion $= a$, that of the fartheſt $= b$: then will the ſum of all the products under each particle into the ſquare of its reſpective diſtance from the axis of motion $= \frac{8\,d\,b\,w + 4\,a^2 w + 3\,w\,r^2}{12}$: by applying this to the braſs cylinder imagined to be inſerted on the ſphere we have $w = 0.0969$, $d = 5.535$, $b = 6.275$, $a = 4.795$, $r = 0.0671$, and the ſum of the products $=$

$$\frac{\overline{8\times 5.535\times 6.275+4\times\overline{4.795}^{2}+3\times\overline{0.0671}^{2}}\times .0969}{12}$$

$= 2.98$, which being ſubtracted from the ſum of the products 249.37, there will remain 246.39, for the ſum of the products of each particle in the ſphere with the cylindrical cavity multiplied into the ſquare of its diſtance from the axis of motion; and the ſum of theſe products in both ſpheres $= 492.78$.

To find the ſum of the products in the rod SQ we have by referring to the rule above, $a = 0$, $d = 3.1375$, $b = 6.275$, $r = 0.0671$, $w = .4017$ oz. and the ſum of the products required $= \overline{\dfrac{8\times 3.1375\times 6.275}{12}+\dfrac{3\times\overline{.0671}^{2}}{12}}\times .4017 = 5.272$, and the ſum for the whole rod $= 10.544$.

The ſum of the products ſimilarly taken in the axis is obtained from the rule contained in page 224, where it is ſhewn to be half the cylinder's weight multiplied into the ſquare of its radius: this will give the ſum of the products in queſtion $= \dfrac{2.4713\times\overline{0.20133}^{2}}{2} = .050$: the ſum of the products therefore in the whole ſyſtem $= 492.78 + 10.544 + .05 = 503.37$, and becauſe the weight of the ſyſtem is 18.244 ounces, the diſtance of the centre of gyration † from the axis will be $= \sqrt{\dfrac{503.37}{18.244}} = 5.2527$. Let this $= r$, and $w = 18.244$, $p = 10$ ounces $=$ the deſcending weight: $q =$ a weight equivalent to the inertia of the wheel FG and the friction wheels when accumulated into the circumference of the wheel FG, this has been || found $= 2.75$ troy ounces, which are equal to 3.017 avoirdupoize ounces uſed in this experiment. Moreover, let d be the radius of the axle $AB = .20133$, and $t = 10$ ſeconds: then the ſpace deſcribed ‡ by the deſcending weight p in ten ſeconds $= \dfrac{lt^{2}d^{2}p}{\overline{p+q}\times d^{2}+wr^{2}} = 15.52$ inches.

† Sect. VI. Prop. VIII.

|| Page 302.

‡ Sect. VI. Prop. XV. Cor. 2.

‡ Sect. VI. Prop. I. Bor. 4.

In ſix ſeconds the ſpace deſcribed from reſt by the deſcending ‡ weight is obſerved = 5.59, which is to 15.52, as the ſquare of 6 to the ſquare of 10; thus it appears by experiment, that the deſcent of *p* is uniformly accelerated, and that the ſpaces deſcribed from reſt are in a duplicate ratio of the times of motion.

IV.

Fig. LXXXIV.

In the ſyſtem A B O Q; let the length of that part of the line which is wound round the axle = 15.5 inches, and let the extremity of it be ſo applied to the axle by a loop and pin, that when p = 10 oz. has during its deſcent deſcribed 15.5 inches, the ſtring may fall off: At this inſtant the force which accelerated the motion of the ſyſtem being removed, the ſyſtem will continue to revolve uniformly, and will perform 24.5 revolutions in 10 ſeconds.

In Prop. XV. Sect. VI. Cor. 1. it is ſhewn that if *p* be the moving force generating angular velocity in any ſyſtem, *d* the radius of the axle to which the moving force is applied, *r* the diſtance of the centre of gyration from the axis of motion, *w* the weight of the ſyſtem, l = 193 inches, and $c = 3.14159$, &c. the weight *p* in deſcending through any ſpace *s*, will generate ſuch an angular velocity in the ſyſtem, as will cauſe it to deſcribe uniformly $\sqrt{\frac{lsp}{c^2d^2p + c^2r^2w}}$ revolutions in a ſecond; provided after *p* has deſcribed the ſpace *s*, the ſyſtem ſuffers neither acceleration nor retardation. Applying this rule to the preſent caſe, we have $s = 15.5$, $p = 10$, $d = .20133$ $w = 18.244$, $r = 5.2527$, and the number of revolutions in one ſecond according to the rule will be =

$$\sqrt{\frac{193 \times 15.5 \times 10}{\overline{3.14159}^2 \times \overline{.20133}^2 \times 10 + \overline{3.14159}^2 \times 18.244 \times \overline{5.2527}^2}}$$

= 2.453, and conſequently in 10 ſeconds the number of revolutions will be 24.53.

V.

Fig. LXXXV.

D C E is a braſs cylindrical plate of which the weight = 188.5 ounces; this cylinder turns horizontally round a fixed vertical axle A B: the axle is ſuſtained in the frame U X Y Z. Motion is communicated to the cylindrical plate by means of a line I G wound round its circumference and from thence going over the wheel F G which reſts on the friction wheels; or by the line L G which is wound round the axle and goes over the ſame wheel F G: the radius of the wheel is to the radius of the axle as 12.6. 1. If the line I G be ſtretched by a weight p = 7 oz. p will deſcribe during its deſcent 51.84 inches in two ſeconds.

The ſum of the ‡ products which are formed by multiplying each particle of the cylinder into the ſquare of its diſtance from the axis of motion is equal to half the cylinder's weight multiplied into the ſquare of its radius, that is in the preſent caſe, the ſum of the prodncts juſt mentioned $= \frac{12.6^2 \times 188.5}{2} = 14963$; and the ſum of the products ſimilarly taken for the axle being .52, the whole ſum will be 14963.52, to this muſt be added the pro-

‡ Page 224.

product of the equivalent weight of the wheel *F G*, and the deſcending weight, multiplied into the ſquare of the wheel's radius, that is, $\overline{3+7} \times \overline{12.6}^2 = 1587$; which will make the ſum of the products for the entire ſyſtem = 16550.52; wherefore the force which accelerates the deſcent of *$p = \frac{\overline{12.6}^2 \times 7}{16550.52} = \frac{1}{14.89}$ and the ſpace deſcribed from reſt in one ſecond $= \frac{193}{14.892} = 12.96$ inches, and in two ſeconds the ſpace will be $12.96 \times 4 = 51.84$, as appear by experiment.

* Sect. VI. Prop. I. Cor. 3.

VI.

Every thing elſe remaining as in the laſt experiment: let the line be wound round the axle, and let a moving force = 10 oz. be applied to ſtretch the line which goes over the wheel L G; the ſpace deſcribed by the deſcending weight in 10 ſeconds will be 12.8 inches.

The ſum of the products formed by multiplying each particle into the ſquare of its diſtance from the axis in the whole ſyſtem being 14963.52; to this add the products formed by multiplying† the equivalent weight of the wheel *FG*, and the deſcending weight into the ſquare of the diſtance at which they act on the ſyſtem from the axis, and ſince that diſtance is 1, the product required will be 13; this being added to the ſum before found, the whole will be 14976.52, and ſince the moving force *p* is 10 oz. the force which accelerates the deſcent of $p = \frac{10}{14976.52} = \frac{1}{1497.652}$. The ſpace therefore deſcribed by *p* during one ſecond from reſt $= \frac{193}{1497.652} = .1288$ ‡ inches and in ten ſeconds the ſpace deſcribed = 12.88 inches.

† Exp. V.

‡ Sect. VI. Prop. I. Cor. 4.

Cor. The diſtance of the centre of gyration from the axis $= \sqrt{\frac{14963.52}{188.5}} = 8.913$, the radius of the axle being $= 1$.

N. B. The aperture in the cylinder which admits the axle is ſo ſmall that whether it is taken into account or not, no alteration is occaſioned in the number ſet down.

VII.

Let A B C repreſent an inclined plane of which the height is to the length as 1 : 16, and let a ſphere deſcend by rolling down this plane: the centre of the ball will deſcribe 34 inches in 2 ſeconds.

Sect. VI. Prop. XIV. Fig. LXXVII.

If the * ſphere's ſurface and the inclined plane were perfectly ſmooth, ſo that the ſphere might ſlide down the plane, the force which accelerated the deſcent would be to the force of gravity as the plane's height to the length, that is, it would be $\frac{1}{16}$ part of the acceleration of gravity; but as the ſphere rolls down the plane, the force of acceleration is diminiſhed in the proportion of 7 : 5, ſo as to become $\frac{5}{7}$ of $\frac{1}{16} = \frac{5}{112}$; wherefore the ſphere's centre will deſcribe by the action of this ‡ force $\frac{193 \times 5}{112}$ or 8.616 inches in one ſecond, and $4 \times 8.616 = 34.4$ in two ſeconds.

* Page 233.

‡ Sect. III. Prop. IV. Cor. 5.

Cor. As all ſpheres, whatever be their magnitudes deſcend by rolling down a given plane with equal abſolute velocities, the angular velocities acquired in a given time will be inverſely as their diameters: ſo that by decreaſing the diameters, this angular velocity might be encreaſed ſine limite, unleſs the effects of friction encreaſed the ſame time, which obſtruct the motion of ſmall ſpheres down inclined planes far more than larger ones.

2. If a cylinder be accelerated along the ſame inclined plane by rolling; in this caſe the force of acceleration being $= \frac{1}{16} \times \frac{2}{3} = \frac{1}{24}$‖, the ſpace deſcribed from reſt in one ſecond $= \frac{193}{24} = 8.041$, and in two ſeconds the ſpace $= 32.164$. ‖ Page 231.

Hence it is manifeſt how much experiment will diſagree with the theory of the deſcent of bodies along planes, unleſs their rotation be taken into account; for if the cylinder were to ſlide without rotation along the plane, during two ſeconds as in the laſt experiment, the ſpace deſcribed would be 48.25 inches, inſtead of 32.165, the difference of which is no leſs than 16.08 inches.

VIII.

Let the ſyſtem repreſented in fig. 84. be removed from the frame; and let two lines be wound round the axle in the manner repreſented in fig. 86. let theſe lines be faſtened to the points L and I at a conſiderable height, L and I being in an horizontal line, and make LT = IH, that the axle may be horizontal. When the ſyſtem is unſuſtained, it will gradually deſcend by unwinding itſelf from the line, and the centre of the axle will deſcribe 28.3 inches in ten ſeconds.

‡ Sect. VI. Prop. XIV. Cyr. 3. Through the ‡ centre of gravity G draw GRO perpendicular to the horizon when the rod OQ is vertical; and let R be the centre of gyration of the ſyſtem when it revolves round the axis AB: aſſume TH in the ſurface of the axle parallel to the axis AB: ſuppoſe the ſyſtem to vibrate as a pendulum on this line, and let O be the cen-

tre of oſcillation; then if the ſyſtem deſcends by unrolling itſelf from the lines *TL*, *IH*, the centre *G* will be accelerated by a force which is that part of the acceleration of gravity, which is expreſſed by the fraction $\frac{SG}{SO}$. In order to find the value of this fraction by theory, we have $SG = .20133$, $GR = 5.2527$, and becauſe $SG : GR :: GR : GO$†, it follows, that $GO = \frac{\overline{5.2527}^2}{.20133} = 137$, wherefore $GO + GS = SO = 137 + .20133 = 137.201$, and $\frac{SG}{SO} = \frac{.20133}{137.201} = \frac{1}{681.4}$, being the force which accelerates the deſcent of *G*: which therefore in ten ſeconds will deſcribe $\frac{193 \times 100}{681.48} = 28.32$ inches.

† Sect. VI. Prop. VI. Cor. 3. and Prop. VIII. Cor. 3.

If the axle *AB* were to be placed between two inclined planes of the ſame heights, lengths and poſitions, and were to deſcend by rolling along them, and the coheſion ſubſiſting between the ſurfaces of the plane and the axle were ſufficient to ſupply the place of the line, by the tenſion of which the rotatory motion was before generated; in this caſe the accelerating force would be equal the force in the laſt example multiplied into the plane's height and divided by its length.

IX.

Every thing elſe remaining as in the laſt experiment, let TL, HI (no longer fixed to L and I.) go round two fixed pullies F and M, and to the extremities let two weights p, p, each of which is equal to 9.108 oz. be ſuſpended: theſe weights will exactly counterbalance the ſyſtem as it is deſcending, and will remain quieſcent.

That moving force which cauſes the rotation of the ſyſtem, or that which ſtretches the ſtrings *LT*, *HI* is that part ‡ of the ſyſtem's weight which is expreſſed by the fraction $\frac{GO}{SO}$, but in the preſent caſe $GO = 137$, and $SO = 137.20133$, and the weight $= 18.244$; wherefore the force which ſtretches the ſtring in exp. viii. $= \frac{18.244 \times 137}{137.2033} = 18.217$, the half of which is 9.108, if therefore each of the weights p be each 9.108, they will together be equivalent to the ſtring's tenſion as the ſyſtem is deſcending, and conſequently will remain quieſcent.

‡ Sect. VI. Prop. XIV. Cor. 3.

Let a ſmall piece of ſteel be fitted to the rod GC, ſo as to admit of being fixed to it at any given diſtance from the centre G, paſſing through the axis of the rod CD: let two edges proceed from the oppoſite ſides of this piece of ſteel in directum in regard to each other, and parallel to the horizontal axis AB: ſo that when the ſyſtem reſts on theſe ſteel edges, the axis of the cylinder DC ſhall be vertical, as repreſented in fig. 86. when motion is communicated to the ſyſtem, it will vibrate as a pendulum on theſe ſteel edges as an axis of motion.

X.

Let the ſteel edges (vid. preceding note) be fixed at .717 parts of an inch from the centre G: on ſetting the pendulum in motion it will be obſerved to vibrate once in each ſecond, provided the arc of vibration be ſmall.

In this caſe $SG = .717$, and by what has preceded $GR = 5.2527$, if therefore O be the centre of oſcillation. we have ‡$GO = \frac{\overline{5.2527}^{2}}{.717} = 38.48$, which being added to $SG = .717$, gives $SO = 39.197$, or nearly 39.2 for the pendulum's length; and a pendulum of this length is known to perform its vibrations in ſeconds.

‡ Sect. VI. Prop. VI.

XI.

Let the ſteel edges be fixed at different diſtances from the centre G: it will be obſerved that the time of the pendulum's vibration will be the leaſt of all when the axis or edges paſs through R the centre of gyration.

This has been demonſtrated in prop. x. ſect. vi. but may be more eaſily ſhewn thus. Let S be the point through which the axis of ſuſpenſion paſſes, O the centre of oſcillation, R the centre of gyration, then in general SG[‡]$: GR :: GR : GO$: now when four quantities are in geometrical [†] proportion, the ſum of the two extremes is always greater than the ſum of the two middle terms, except when the four terms are all equal: therefore $SG + GO$ is always greater than $2GR$, except when $SG = GR = GO$; $SG + GO$ therefore will be the leaſt poſſible when $SG = GR$, that is, when the point S through which the axis of ſuſpenſion before coincides with R the centre of gyration.

[‡] Sect. VI. Prop. VI. Cor. 3. and Prop. VIII. Cor. 3.

[†] Eucild. El. Lib. V. Prop. ult.

Cor. When a given body vibrates in the leaſt time poſſible, the plane of vibration being the ſame, the diſtance between the axis of ſuſpenſion and centre of gravity is equal to the diſtance between the centre of gravity and the centre of oſcillation, for by what has preceded $SG = GR$, and $GO = \frac{GR^2}{SG} = GR = GS$.

It muſt be obſerved concerning the experiments made on bodies which revolve round axes, that the friction is diminiſhed, or indeed almoſt wholly removed as to any ſenſible effects three ways: 1. by making the axle revolve horizontally on friction wheels: 2dly, by placing the axle vertical, the lower point of the axis being ſuſtained in a very ſmall and well poliſhed cavity, as in fig. 84. here the ſmallneſs of the ſurfaces in contact has an effect in removing tenacity, and we may ſay friction too, notwithſtanding ſome experiments to the contrary equal to that produced by the friction wheels: 3dly, when the ſyſtem vibrates; by ſuſpending it on ſharp edges as in the latter experiment.

SECT. IX.

ON THE MOMENTUM † OF BODIES IN MOTION.

FORCE has been * defined that which cauſes a change in the ſtate of motion or quieſcence of ſubſtances: The action of animal muſcles, the impact of bodies either ſolid or fluid, attractions and repulſions of various ſorts have been enumerated as inſtances of this motive power. The Idea of force is originally acquired from the ſenſe of feeling, and through the medium of that ſenſe by the eye alſo; but this idea extends only to the degree of force, not to the quantity or any meaſure of it. Thus when muſcular force is exerted in order to overcome reſiſtances of any kind, there is no difficulty in determining when the force is greater or leſſer, but as it is not in itſelf a mathematical quantity, the proportion between the greater and leſſer, and the exact increaſe or diminution of its intenſity cannot be known except by applying either ‡ ſpace or number as meaſures of it.

Application of mathematics.

This application is effected by the axioms or laws of motion, which may be term-

* Page 1.

‡ Sect. II. Prop. I.

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† The expreſſions, quantity of motion, momentum, and moment, are uſed ſynonymouſly throughout this ſection.

termed propoſitions intermediate between Geometry and Philoſophy; through theſe, mechanics becomes a mathematical branch of phyſics, and its concluſions poſſeſſed of ſuch coherence and conſiſtency among themſelves and with matter of fact, as are rarely to be found in other [a] branches, which admit not of ſo intimate an union with the ſcience of quantity.

Laws of motion not demonſtrated.

Since the Newtonian laws apply mathematical relations to the motion of bodies, it may be expected that the truth of them ſhould reſt on mathematical proof: but it is to be remembered, that theſe laws are aſſumed in mechanics as axioms are in geometry; and an axiom is underſtood to be a propoſition which either from being intuitive requires not, or from the nature of the objects to which it is applied is incapable of, demonſtrative evidence.

This is the caſe with the laws of motion: They cannot be deduced from any reaſoning à priori, for we know nothing concerning matter but from experience; and though experiment confirms the truth of them in a variety of inſtances, and to a certain degree of exactneſs, yet be-

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[a] Chemiſtry and Electricity are of this ſort: the principles of theſe are arranged ſyſtematically, and are referable to a few general laws; but they are not univerſally and without exception conſiſtent among themſelves and with experience, as the principles of mechanics are.

between that degree and the mathematical relation expreſſed in the laws, there may be infinite varieties of different relations which experiment cannot diſcover.

Laws of motion collected from various arguments.

[b] If then it be enquired, from what evidence our aſſent to theſe axioms is derived, it may be replied, the evidences are of various kinds. 1. [c] From the conſtant obſervation of our ſenſes, which tend to ſuggeſt the truth of them in the ordinary motion of bodies, as far as the experience of mankind extends. 2. From [d] experiments, properly ſo called: In theſe it always appears, that when the moſt effectual means are uſed in order to ſubject theſe axioms or laws of motion to the ſevereſt and moſt minute examinations, by removing all impediments which may tend to occaſion any deviation from the truth, the reſult is a more near agreement between the experiments and the axioms juſt mentioned, the more perfectly the intentions of the experiments are accompliſhed. This will lead to

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[b] The laws of motion not admitting of demonſtration, the truth of them muſt be collected from the beſt evidence the nature of the ſubject will admit.

[c] This argument depends not on the accuracy of theſe common obſervations, but from their extent; nothing contrary to the axioms being obſerved in the great variety of motions which are the daily objects of the ſenſes.

[d] In experiments every power is removed as far as may be, except thoſe which are the objects of examination.

a belief, that if it were poſſible to remove every imperfection from the conſtruction of experiments, and to increaſe the acuteneſs of the ſenſes, of which they are the objects, in an infinite degree, a mathematical coincidence between ſuch experiments and the laws of motion would be obſerved.

3. From arguments à poſteriori: — Let a propoſition be aſſumed as true even without evidence of any kind; if by ſtrict and logical reaſoning, various concluſions are deduced, which upon examination are found conſiſtent among themſelves, and with experience, this will be a preſumptive proof in favour of the principle aſſumed; and our aſſent to it will be the more ſtrongly enforced in proportion as the concluſions inferred and the compariſon of them with experience have been more extenſive. From the Newtonian axioms aſſumed as true, a ſyſtem has been deduced and compared with phenomena in numberleſs caſes: it has been applied to the motion of the planets and comets, to that of bodies on the earth's ſurface; even to the motion of thoſe minute particles which compoſe both ſolid and fluid ſubſtances: A perfect agreement between theſe conſequences of the axioms and matter of fact has been the reſult, no one inſtance excepted. Theſe and other ſimilar ar-

arguments, upon the whole amounting to evidence ſcarcely inferior to mathematical demonſtration, are the grounds on which the laws of motion, although incapable of direct proof, are received as axioms, from which the various theorems concerning the effects of forces are ſyſtematically demonſtrated.

In treating of mechanical as well as other ſubjects, it is certainly allowable to deſcribe terms by any form of words that may ſeem beſt adapted to the purpoſe of conveying a diſtinct and adequate idea of the object deſcribed. [e] The meaſure of the quantities of motion impreſſed on bodies, or deſtroyed in them, may be defined by the joint ratios of their quantities of matter and velocities, or of the quantities of matter and ſquares of their velocities: but as definitions affirm nothing, being in themſelves incapable of expreſſing truth or falſity, if either of theſe meaſures were ſaid to be right excluſively of the other, this would be only miſapplying the definition by converting it into an affirmation or propoſition. The ſame objection may be urged againſt many arguments which have been brought both from theory and experiment in the way of proof, in order to eſtabliſh one of theſe meaſures, or controvert the other. The following are of this kind, and when cleared from the multitude of words which controverſy has heaped upon them may be deſcribed thus. Let the meaſure expreſſed [f] by $\frac{V^2}{v^2} \times \frac{Q}{q}$ be aſſumed as that by which the quantity of motion is defined: becauſe the total effects of bodies in motion eſtimated by the reſiſtances which they overcome, and the whole ſpaces deſcribed during their conſtant retardation jointly, are obſerved in experiments to be always proportional to [g] $\frac{V^2}{v^2} \times \frac{Q}{q}$, or in other expreſſions, becauſe $\frac{V^2}{v^2} \times \frac{Q}{q} = \frac{M}{m} \times \frac{S}{s}$‡, either $\frac{V^2}{v^2} \times \frac{Q}{q}$ or $\frac{M}{m} \times \frac{S}{s}$ was concluded[h] to be the [i] true meaſure

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[e] The definition does not imply the deſcription of any properties of motion, but only of the meaſure of its quantity. By the ratio of the quantities of motion in any two moving bodies, is defined to be meant, the joint ratios of the velocities and quantities of matter contained in the bodies: it is therefore impoſſible to miſtake what is meant by the expreſſion, quantity of motion, according to this deſcription; and this is the only object of the definition: — the ſame may be applied to any other definition of the moments of bodies.

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[f] Whenever any meaſure of the quantity of motion is mentioned, it implies the conſideration of two bodies, the moments of which are compared: let the two bodies be Q and q, and their reſpective velocities V and v: then the joint ratios of their quantities of matter and velocities will be $V \times Q$ to $v \times q$, or according to the notation uſed in ſect. ii. $\frac{V}{v} \times \frac{Q}{q}$ to 1. In the following pages when $\frac{V}{v} \times \frac{Q}{q}$ is mentioned as expreſſing the quantities of bodies' moments, it means that the quantity in the body Q is to that in q as $\frac{V}{v} \times \frac{Q}{q}$ to 1, or as $V \times Q$ to $v \times q$: a ſimilar conſtruction is to be underſtood concerning the other meaſure $\frac{V^2}{v^2} \times \frac{Q}{q}$.

‡ Page 35. 261. Sect. VII. Exp. XII. page 333.

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[g] The notation uſed in prop. v. and vi. ſect. iii. is adopted in this ſection, and the ſignification of the quantities is the ſame as deſcribed in thoſe propoſitions.

[h] The quantity of rectilinear motion any how eſtimated is referred to the ſame direction: two equal quantities of motion acting in oppoſite directions exactly counteract each other, and vice verſa, if two moments acting in contrary directions counteract each other, they are inferred to be equal. It is needleſs to mention caſes of oblique forces, ſince nothing is gained or loſt in a given direction by the reſolution or compoſition of forces.

[i] A practical inſtance or two in which both meaſures of the quantities of motion are applied, may be here inſerted. 1. When piles are driven into the earth by the percuſſion of heavy bodies impinging on them, the ſpaces they deſcribe at each impact of the weight are eſtimated from the theory of Leibnitz, &c. thus. Let the force with which the earth reſiſts the entrance of the piles, be to the weight of the impinging body as $m : M$; and let the ſpace through which the pile is driven, be to the ſpace through which the heavy body falls as $S : s$; then, as the whole motion of the impinging weight is deſtroyed by reſiſtance in the oppoſite direction, the moments of both are equal and contrary: in the equation $\frac{V^2}{v^2} \times \frac{Q}{q} = \frac{M}{m} \times \frac{S}{s}$, ſince $V^2 Q = v^2 q$ by the preſent caſe, it follows that $M \times S = m \times s$: that is, the impinging weight multiplied into the height from

ſure of the quantities of motion deſtroyed. This is only one out of a many inſtances of ſimilar experiments made on bodies both accelerated and retarded, which have been brought to eſtabliſh this meaſure of momentum. It is however evident, that nothing is proved by theſe experiments, except the truth of the mechanical propoſition expreſſed in the equation $\frac{V^2}{v^2} \times \frac{Q}{q} = \frac{M}{m} \times \frac{S}{s}$, § which, without any reference to experiments, is deduced from the laws of motion.

§ Sect. III. Prop. VI.

Exactly in the ſame manner experiments ſhew, that * $\frac{V}{v} \times \frac{Q}{q} = \frac{M}{m} \times \frac{T}{t}$, from which

* Sect. VII. Exp. VI. & VII. & VIII.

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from which it falls, is equal to the reſiſtance exerted againſt the pile's entrance, multiplied into the ſpace to which it is driven. This, it is evident, holds true, whatever be the reſiſtance which is oppoſed to the pile, becauſe while it moves through a ſmall ſpace the reſiſtance is conſtant, and its proportion to the impinging weight muſt be always expreſſed in the rule by the ratio $m : M$.

2. Let two boats of different weights be connected together by a line, and let the line be ſtretched by pulling the larger veſſel toward the ſmaller, or the ſmaller toward the larger; applying the equation $\frac{V}{v} \times \frac{Q}{q} = \frac{M}{m} \times \frac{T}{t}$, $M = m$, becauſe the tenſion of the rope acts equally on each body: and $T = t$, becauſe they begin and continue to move together; wherefore, becauſe $MT = mt$, it follows, that $V \times Q = v \times q$, or the velocities of the two boats, at any given inſtant, will be in the inverſe proportion of their quantities of matter, the reſiſtance of the water not being conſidered.

which it may with equal reaſon be concluded, that either $\frac{V}{v} \times \frac{Q}{q}$ or $\frac{M}{m} \times \frac{T}{t}$ is the true meaſure of the quantities of motion in bodies.

Theſe arguments, however inconcluſive in themſelves[k], being extended and diverſified, gave riſe to others leſs ambiguous: whatever difference of opinion might ſubſiſt concerning the meaſures in queſtion, it was agreed univerſally, that the quantity of motion, if meaſurable at all, ſhould be proportional to the cauſes by which it was generated, or to the effects produced by it, eſtimated according to the ſame meaſure; and that a conſtant equality between the quantity of motion, before and after its communication, ſhould be preſerved in bodies of every kind and any how moved.

This principle applied to the [l] meaſure which eſtimates the motion of bodies by their quantities of matter into the ſquares of their velocities, Bernoulli denominated virium vivarum conſervatio, imagining any given moment thus meaſured to be univerſally permanent and immutable in quantity: † " Hinc patet vim vivam eſſe " aliquid reale et ſubſtantiale quod per " ſe ſubſiſtit, &c. —— unde concludimus " quamlibet vim vivam habere ſuam de- " terminatam quantitatem de quâ nihil " perire poteſt quod non in effectu edito " reperiatur."

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[k] The experiments firſt made on the effects of percuſſion on ſoft bodies, &c. in order to prove the truth of the meaſure of momentum expreſſed by $\frac{V^2}{v^2} \times \frac{Q}{q}$, were prior in time to Bernoulli's theory denominated Conſervatio virium vivarum.

[l] Bernoulli termed this principle virium vivarum conſervatio when applied to the meaſure $\frac{V^2}{v^2} \times \frac{Q}{q}$ only; but motus being the word uſed to imply quantity of motion in general, conſervatio motus will expreſs the application of the ſame principle, to the motion of bodies eſtimated as to quantity by either of the meaſures.

† Bernoulli, Vol. III. De Vera Notione Virium vivarum.

Perceiving his principle to obtain in ſeveral inſtances, he too haſtily concluded it to be univerſal; and, as is well known, by reaſoning from it [m] ſyſtematically, was led into miſtakes and inconſiſtences, which he choſe to miſemploy his ingenuity in defending, rather than forego a preconceived theory. If indeed the nature of motion were ſuch, that a given quantity of it, any how eſtimated remained in all caſes permanent, ſuffering neither augmentation nor diminution in its communication, this immutable quality would in itſelf conſtitute the reality and truth of the meaſure thus ap-

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[m] In ſome *caſes he refers the effects of forces to directions different from thoſe in which the forces were impreſſed, and thence deduces concluſions favourable to his theory, but as is well known contrary to the laws of motion.

* Diſcours ſur le Mouvement, Vol. III.

applied; but not excluſively of other meaſures with which (if poſſible) the general principle “ conſervatio motus” were conſiſtent.

To conſider this more particularly, let either of the meaſures, that of Bernoulli for example, be aſſumed. It is plain, that unleſs the permanency in a given quantity of motion thus expreſſed, were demonſtrated in all kinds of bodies any how put in motion, or unleſs the truth of this principle were collected by induction ſufficiently extenſive, and contradicted in no one inſtance, it ſhould rather be accounted an individual property of motion, than a general law: but to give the argument its due force, let it be ſuppoſed that the “ conſervatio motus” eſtimated by $\frac{V^2}{v^2} \times \frac{Q}{q}$ is obtained in general, without limitation or exception; ſtill if the ſame univerſality could be proved to belong to the permanency of motion as expreſſed by $\frac{V}{v} \times \frac{Q}{q}$, there ſeems no reaſon for aſſuming one of them as the true meaſure of the moments of bodies in motion rather than the other. But the truth is, the principle obtains not according to either of the meaſures, except in particular caſes, which may be demonſtrated as the other properties of forces are from the general laws or axioms.

In the rectilinear motion of bodies, accelerated from quieſcence or retarded § until they are at reſt, the permanency of any given quantity of † motion is demonſtrated from the axioms, whether that motion be eſtimated by one meaſure or by the other. § Note h p. 362. † Sect. III. Prop. VI. and VII.

In bodies which revolve round fixed axes, the principle obtains without exception when the moment is meaſured by the quantity of ‖ matter into the ſquare of the velocity, but fails when meaſured by the quantity of [n] matter into the velocity; a given quantity of motion thus eſtimated being alterable in any aſſigned ratio. ‖ Sect. VI. Prop. XVII.

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[n] This may be deduced from * Cor. 7. Prop. 16. If it were true that a given quantity of motion, eſtimated by the meaſure $\frac{V}{v} \times \frac{Q}{q}$, is permanent, it would follow that any force p acting for a given time t, muſt produce ſuch a velocity in a revolving ſyſtem, that the ſum of the products formed by multiplying each particle into its velocity, ſhall be the ſame as the body p multiplied into the velocity which it would acquire in falling freely by its gravity during the time t: but the contrary is eaſily ſhewn. Let the whole maſs in the ſyſtem ABC be collected into R: then the notation of ‡ prop. xvi. remaining, the velocity generated in R in the time $t = \frac{2ltdp}{wr}$, which being multiplied into the maſs w will give $\frac{2ltdp}{r}$ for the product under the quantity of matter and into its velocity: but the velocity generated in p when it falls freely for t ſeconds is $= 2lt$†, which being multiplied into p gives $2ltp$ for the product of the maſs into the velocity acquired, which it is plain may differ from the former in any ratio. * Sect. VI. Fig. LIX. ‡ Sect. VI Cor. 7. † Sect. VI. Prop. II.

In the communication of motion to bodies by colliſion, when the direction of the ſtroke paſſes through the centres of gravity, the principle in queſtion holds univerſally according to the meaſure $\frac{V}{v} \times \frac{Q}{q}$, but fails when the moments are eſtimated by $\frac{V^2}{v^2} \times \frac{Q}{q}$ in every caſe, except when both bodies are perfectly elaſtic, or one perfectly elaſtic and the other perfectly hard.

Laſtly[o], when motion is communicated to bodies by impact, the direction of which paſſes not through the centres of gravity, the quantity of motion communicated, whether eſtimated by one meaſure or the other, preſerves neither equality nor any conſtant proportion to the quantity of motion impreſſed: theſe conſiderations make it evident, that from the fundamental properties of forces as expreſſed in the general laws or axioms, which are controverted by no one, any given quantity of a body's motion, as meaſured by the ſum of the products formed by multiplying each particle into the velocity, or into the ſquare of the velocity;

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[o] This is an argument againſt the permanency of a given quantity of motion according to any meaſure: if a body A impinge directly with the velocity V on a quieſcent body or ſyſtem of bodies B, in the direction of a line which joins their centres of gravity, and the bodies ſhould be nonelaſtic, the whole maſs will go on after the impact with the velocity $\frac{AV}{A+B}$, or if the inertia of A be incomparably ſmaller than that of B, the common velocity will be $\frac{AV}{B}$. If the impact be ſtill perpendicular to the plane which it ſtrikes, although the direction of it paſſes not through the centres of gravity, the velocity of the centre of gravity will be the ſame as before, i. e. $\frac{AV}{B}$, and at the ſame time the ſyſtem will revolve round that centre with different degrees of angular velocity, according as the direction of the impact paſſed nearer to the centre of gravity, or farther from it, as is demonſtrated in ſect. x. of this book: it is alſo deduced from different principles in a

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a very ingenious * paper on the ſubject of rotation by the Rev. Mr. Vince of the univerſity of Cambridge, to whom the prize medal of Sir G. Copley, for the year 1780, was adjudged by the preſident and council of the Royal Society.

When bodies revolve round a progreſſive axis of motion, paſſing through the center of gravity, the ſum of the moments, if they were eſtimated in a given direction, as far as regards the rotation, would be $=0$. This would alſo be true when the axis is fixed; by which method of reaſoning, the moment of every body revolving round a fixed axis, which paſſes through the centre of gravity would be nothing; whereas according to any ſuppoſed meaſure, the quantities of motion impreſſed on revolving bodies, are not only finite, but are in aſſignable proportions to each other: this being granted, when any quantity of matter revolves in free ſpace, by the impulſe of a body impinging on it with a given velocity, the quantity of motion in the impinging body remaining the ſame, any different quantities of motion may be produced in the body ſtruck, according as the impact is made nearer to the centre of gravity, or further from it; a concluſion which is contrary to the hypotheſis of moment according to any meaſure.

* Philoſ. Tranſ. for 1780, p. 550.

locity, is not of that permanent and immutable nature, which if really exiſting, would be the only ground on which any meaſure of ſuch quantity could be eſtabliſhed. This being the caſe, all controverſy concerning which of the expreſſions, i. e. that of $\frac{V^2}{v^2} \times \frac{Q}{q}$, or $\frac{V}{v} \times \frac{Q}{q}$ is the true meaſure of the quantities of motion in bodies, muſt be regarded as verbal, i. e. a diſpute about a term or definition, except only as far as relates to the aſſumed
* Page 364. note l. principle, denominated * "conſervatio motûs," which implies the quality of the ſums of products formed by multiplying each particle of a body into ſome power of its velocity: this being a diſtinct propoſition admits of ſuch examination as is neceſſary to confirm or diſprove it.

It follows from what has preceded, that the permanency of motion, during its communication, ſhould not be applied in the demonſtration of mechanical propoſitions, becauſe in ſyſtematic reaſoning, no propoſition is to be aſſumed as true, except the axioms or fundamental laws, and ſuch as have been regularly inferred from them. Thus in demonſtrating the laws of colliſion, the equality of moments before and after the ſtroke, as eſtimated by any meaſure, ſhould not be aſſumed as a ſtep in the demonſtration, becauſe nothing is mentioned concerning the equality of theſe moments in the laws of motion: the equality of moments according to the meaſure $\frac{V}{v} \times \frac{Q}{q}$ can indeed be deduced from the axioms, which deduction is the ſame as the demonſtration of the laws of colliſion.

Let *A* and *B* be the centres of gravity of two bodies moving in free ſpace, and let *A* be ſuppoſed to act on *B* in the direction *BD* with any kind of force variable in any manner; then will *B* react on *A* in the direction *AC* with equal force; and if the time in which the effects of theſe forces are produced, be aſſumed evaneſcent, the force will for that inſtant be conſtant. In the corollary to prop. I. ſect. III. it appears, that in any given particle of time, the variation of velocity generated in that time will be always proportional to the accelerating force, that is, to the moving force directly, and quantity of matter inverſely; or, in the preſent caſe, ſince the force with which *A* impels *B* is equal to that with which *B* reacts on *A*, the velocities generated in theſe bodies during the ſame time, will be inverſely as the quantities of matter contained in them: any velocity Fig. LXXVII. III. Law of motion.

generated in *A* in the direction *AC* being conſidered as a decrement of its velocity referred to the direction *AD*.

When a ſphere *A* impinges directly on any other ſphere *B*, moving in the ſame direction, the acceleration of *B* and retardation of *A* will be effected gradually during the time in which the figures of the two bodies are changed by the impact; while therefore the centre of *A* moves with a velocity greater than that of *B*, ſo long will *B* continue to gain and *A* to loſe motion, until the change of their figures is the greateſt, the centres being then neareſt to each other; at this inſtant they will begin to go on with the ſame velocity, in bodies of every kind of texture. If the bodies be ſuch as poſſeſs a power of reſtoring the figures which have been thus changed, this power alſo will act equally on each body in contrary directions, for the ſame time, and will create the greater augmentation of *B*'s motion, and the greater diminution of *A*'s, according to the proportion of the reſtitutive power to that whereby the bodies reſiſted the force which effected the change of their figures: this reſtitutive power in bodies perfectly void of elaſticity being nothing. Suppoſe the bodies *A* and *B* to be of this ſort, and to move before the impact with the velocities *a* and *b*: let

III. Law of motion.

let the time in which the change of figure is effected be divided into innumerable inſtants. Suppoſe during the firſt of theſe inſtants, that is, at the firſt contact between *A* and *B*, *A* loſes any ſmall part of its velocity repreſented by p, and during the ſame inſtant that *B* gains an increment of velocity repreſented by u; in the next inſtant let q be the loſs of *A*'s velocity, and w the gain of *B*'s, and ſo on during innumerable inſtants, let p, q, r, s, &c. be the ſucceſſive values of the loſs of *A*'s velocity, and u, w, t, z, &c. the contemporary increments of the velocity of *B*: becauſe each of theſe inſtants is evaneſcent, the force which urges the two bodies in contrary directions will be conſtant in any given inſtant, and acting for the ſame time will generate increments or decrements of velocity which are inverſely proportional to the *quantities of matter contained in *A* and *B*; for this reaſon in the firſt inſtant we ſhall have $p : u :: B : A$; moreover in the ſecond inſtant $q : w :: B : A$, and ſo on: ‡wherefore $p + q + r + s$, &c. : $u + w + t + z$, &c. as $B : A$; and if $x = p + q + r + s$, &c. repreſents the whole loſs of *A*'s velocity, and $y = u + w + t + z$, &c. the total increaſe in that of *B*, the proportion will be as $x : y :: B : A$: but when $x =$ the whole ſum of $p + q + r + s$, &c. and $y =$ the other

* Page 371.

‡ Euclid. El. Lib. V. Prop. XII.

other correſponding ſum, the bodies begin to move with a common velocity, and ſince a and b were the reſpective velocities before the impact, it will follow that after the impact $a - x = b + y$, or $x + y$ ‖ Page 373. $= a - b$: that is, ſince‖ $x = \frac{By}{A}$, $\frac{By + Ay}{A} = a - b$, and $y = \frac{\overline{a - b} \times A}{A + B}$, which gives this analogy, as $A + B : A :: a - b :$ to y, $=$ the velocity gained by B: if therefore we add to $\frac{\overline{a - b} \times A}{A + B}$, the original velocity of B which was b, the ſum will be the velocity of B after the ſtroke $= \frac{Aa - Ab}{A + B} + b = \frac{Aa + Bb}{A + B}$, or the common velocity, with which the two bodies begin to move when their change of figure is the greateſt, whatever be their texture.

Cor. Multiply $\frac{Aa + Bb}{A + B}$ the common velocity by the ſum of the bodies: the product will be $= Aa + Bb$, which is the ſame as the ſum of the products of each body into its reſpective velocity before the ſtroke.

When the bodies are poſſeſſed of a power by which they endeavour to reſtore the change of figure effected by the impact, this power may act with different degrees of intenſity, and for different times according to the textures of the bodies: but the time of its action, whatever be the quantity of it, may be divided into innumerable inſtants, correſponding to thoſe into which the time of effecting the greateſt change was divided. Now ſuppoſe the reſtitutive power ſuch as will generate in the body ſtruck, during any inſtant, a velocity which is to the velocity before generated, while the figures were changing in the correſponding inſtant, as m to 1: m being aſſumed as a meaſure of the elaſtic force: then during the ſucceſſive inſtants in which the changed figures are reſtored, the diminution of A's velocity will be $m \times \overline{p+q+r+s}$, &c. $= mx$, and $m \times \overline{u+w+t+z}$, &c. $= my$ the cotemporary increaſe of B's velocity: and ſince before the elaſtic power took place A's velocity was $a - x$, and B's $= b + y$, after that power has ceaſed to act A's velocity muſt be $= a - x - mx$, and B's velocity $= b + y + my$, or if $m + 1$ be put $= n$ the velocity of A after the impact will be $a - nx$, and that of $B = b + ny$; but $x = \frac{By}{A}$ ‡ in general; this will give the velocity of A after the ſtroke $= \frac{Aa - Bny}{A}$ and the velocity of $B = b + ny$.

Cor.

‡ Page 373.

Cor. Multiply each body after the ſtroke into its reſpective velocity, the products will be $Aa - Bny + Bb + Bny$, the ſum of which is $Aa + Bb$, the ſame as the ſum of the products formed by multiplying each body into its velocity before the ſtroke: if therefore the quantity of motion be defined by the ſum of theſe products, it will follow, that in the colliſion of all kinds of bodies, the quantity of motion is not altered by the effects of the impact.

If the elaſtic power be equal to that which reſiſts the change of figure, m will be $= 1$, and $m + 1 = n = 2$; in this caſe therefore A's velocity after the impact will be $a - 2x$, that of $B = b + 2y$; ‡ Page 374. and ‡ becauſe $x + y = a - b$, by ſubſtitution, the difference of the velocities after the ſtroke will be $= a - b - 2a + 2b = b - a$, the ſame as the difference of velocities before the ſtroke, only negative in reſpect of it. From hence follows the propoſition ſo much inſiſted on by Bernouilli; that in the impact of perfectly elaſtic bodies, the ſum of the products formed by multiplying each body into the ſquare of its velocity is not altered by the impact.

His demonſtration is this: The other notation remaining, let p be the velocity of A and q that of B after the ſtroke; then ſince by what has † preceded $Aa + Bb = Ap + Bq$, or $^{*}Aa - Ap = Bq - Bb$,

† Supra.

* Diſcours ſur le Mouvement. Vol. III.

Bb, [*] and $a-b=q-p$, or $a+p=q+b$: by multiplying $Aa-Ap$ into $a+p$, and $Bq-Bb$ into $q+b$, we have $Aa^2-Aap+Aap-Ap^2=Bq^2-Bqb+Bqb-Bb^2$, or $Aa^2+Bb^2=Ap^2+Bq^2$, which is the equality affirmed: a geometrical demonſtration is given of the ſame propoſition by Maclaurin [†].

From the [‡] laws of colliſion it appears, that the quantity of motion as meaſured by the ſum of the products expreſſed by $V\times \mathcal{Q}$, is not altered by the colliſion of bodies of any kind: but the equality of the quantities of motion, before and after the impact, according to the meaſure expreſſed by $V^2\times \mathcal{Q}$, obtains only in thoſe bodies which are perfectly elaſtic.

It is extraordinary that Bernoulli ſhould inſiſt upon the demonſtration of this particular property of perfectly elaſtic bodies, in order to confirm an hypotheſis which he aſſumed as a general law of motion, and conſequently ought to extend to bodies of every kind; had his proceſs been inverted, more general concluſions would have been the reſult. Inſtead of demonſtrating the equality of the ſums of the products formed by multiplying each particle into the ſquare of its velocity before and after the impact, on a ſuppoſition that the elaſtic force is equal to that which reſiſts the change

[*] Supra,

[†] View of Sir J. Newt. Phil. Book II. Chap. IV.

[‡] Page 376.

of figure, it may be more expedient to inquire, having given the ſums of the products expreſſed by the maſs multiplied into any power of the velocity as V^n, what muſt be the elaſtic force, in order that the ſums of the products, before and after the ſtroke, ſhall be equal. Let the notation uſed above be reſumed; then if u be put to denote the reſtitutive power when 1 is that which reſiſts the change of figure, and let $1 + u = s$, we ſhall have the velocity of A after the ſtroke $= a - sx$, or ſince ‡ $x = \frac{By}{A}$, the velocity of $A = a - \frac{Bsy}{A}$, and that of $B = b + sy$; and the ſum of the products formed by multiplying each maſs into the n^{th} power of its velocity after the ſtroke will be $\frac{\overline{Aa - Bsy}^{\,n}}{A^{n-1}} + B \times \overline{b + sy}^{\,n}$, which by the problem is equal to $Aa^n + Bb^n$: by ſolving the equation $\frac{\overline{Aa - Bsy}^{\,n}}{A^{n-1}} + B \times \overline{b + sy}^{\,n} = Aa^n + Bb^n$, the value of s will give that of u, or the reſtitutive power ſought. Suppoſe for example, it were required to aſſign u when the ſums of the products formed by multiplying the maſs into the ſquare of the velocity are equal before and after the impact:

‡ Page 373.

pact: here $n = 2$, which being ſubſtituted, the equation will become $A^2a^2 - 2ABasy + B^2s^2y^2 + ABb^2 + 2BAbsy + ABs^2y^2 = A^2a^2 + BAb^2$, or $s^2 \times \overline{ABy^2 + B^2y^2} - s \times 2ABy \times \overline{a - b} = 0$, or ſince $y = \frac{\overline{a - b} \times A}{A + B}$, the equation will be $s \times \frac{\overline{A + B} \times \overline{a - b} \times A}{A + B} = 2A \times \overline{a - b}$, and $s = 2$; wherefore $u = s - 1 = 1$; that is, the force of elaſticity muſt be equal to that which reſiſts the change of figure. Suppoſe $n = 3$, or, let it be required to aſſign the elaſtic force, ſo that the ſums of the products under the maſs and cube of the velocity ſhall be equal before and after the impact: in the general equation

$$\frac{\overline{Aa - Bsy}^{\,n}}{A^{n-1}} + B \times \overline{b + sy}^{\,n} = Aa^n + Bb^n,$$

ſubſtitute 3 for n, and $\frac{A \times \overline{a - b}}{A + B}$ for y; the equation will become $s^2 + s \times \frac{3Ba + 3Ab}{\overline{A - B} \times \overline{a - b}} = \frac{\overline{3A + 3B} \times \overline{a + b}}{\overline{A - B} \times \overline{a - b}}$, from which s is eaſily obtained. It is manifeſt here, that the elaſtic power to ſatisfy the conditions of the problem, will depend on the quantities of matter of the bodies, and on the relative

tive velocities before the ſtroke: if $b = 0$, that is, if the body ſtruck ſhould be quieſcent before the impact, s will be found =

$$\sqrt{\frac{9B^2}{4\times\overline{A-B}^2} + \frac{3A+3B}{A-B}} - \frac{3B}{2\times\overline{A-B}},$$

in which caſe it appears, that the quantity of elaſticity neceſſary to ſatisfy the conditions of the problem, will now not at all depend on the velocity of the impinging body, but on the maſs in the bodies only: if $A = 2$ and $B = 1$, s will be = 1.85410, and the reſtitutive force $u = s - 1 = .85410$; according therefore to the conditions mentioned above, if the force whereby the ſpheres A and B reſtore the change of figure be to the force which reſiſted that change in the proportion of .85410 to 1; the ſums of the products of each particle into the cube of its velocity, before and after the impact, will be equal: ſo that if the quantities of motion in bodies were defined to be meaſured by the maſs into the cube of the velocity, the ſum of the moments before and after the impact would be equal in this particular caſe.

It is not probable that the theory of motion, however inconteſtable its principles may be, can afford much aſſiſtance to the practical mechanic; and there appears as little reaſon to imagine that any errors or miſconceptions which may have been propagated concerning the effects of forces conſidered in a theoretical view, have at all impeded the due conſtruction of uſeful machines, ſuch as are impelled by the force of wind or water, by ſprings, or any other kind of motive power. Machines of this ſort, owe their origin and improvement to other ſources: it is from long experience of repeated trials, errors, deliberations, corrections, continued throughout the lives of individuals, and by ſucceſſive generations of them, that the ſciences ſtrictly called practical, derive their gradual advancement from feeble and aukward beginnings, to their moſt perfect ſtate of excellence.

Thoſe however, who are the moſt converſant in the conſtruction of machines, and to whom the manner of their operation is moſt familiar, will naturally be inclined to inquire into the reaſons or cauſes of motions thus obſerved, and to compare them with the rules which are deduced from the general and undiſputed principles of mechanics; but it is to be apprehended, that the obſtacles which are occaſioned by friction, tenacity, the irregular action of the impelling force, and of the reſiſtances, which ſcarcely admit of preciſe eſtimation, muſt greatly obſtruct, if not wholly prevent, a ſatisfactory and accurate compariſon of the theory with practice, in any ordinary way of conſidering the ſubject. Theſe difficulties will appear to be greater, from conſidering, that the motion of machines which are impelled by wind or by water is uniform (ſetting aſide the irregularities of the moving power, &c. which affect not the preſent argument) from whence it is known that in theſe machines, when there is no weight which acts againſt the moving force, the reſiſtance of friction muſt be equal to the whole moving power, whereas in the theory of mechanics, friction is conſidered as having no effect.

Mr. Smeaton in his paper on mechanic power (publiſhed in the Philoſophical Tranſactions for the year 1776) allows, that the theory uſually given, will not correſpond with matter of fact, when compared with the motion of machines, and ſeems to attribute this diſagreement rather to ſome deficiency in the theory, than to the obſtacles which have prevented the application of it, to the complicated motion of engines, &c. In order to ſatisfy himſelf, concerning the reaſons of this diſagreement, he conſtructed a ſet of experiments, which, from the known abilities and ingenuity of the author, certainly deſerve great conſideration and attention from every one, who is intereſted in theſe inquiries: and if any doubt could be entertained concerning the truth of the theory, as deduced from the Newtonian laws of motion, theſe experiments, among others, might be aſſumed as ſtandards, to which its examination might be referred: but as it muſt be confeſſed, that the evidence upon which the certainty of this theory reſts, is ſcarcely leſs than mathematical, it will be more eligible to refer mechanical experiments of every kind to the theory, as a means of diſcovering how far, the unavoidable imperfections of conſtruction and obſervation, the reſiſtance of friction, &c. may have cauſed them to deviate from the truth. The ingenious author himſelf, has afforded us ſufficient means for this purpoſe, by giving the exact weights and dimenſions of the parts of his inſtrument; the outline of which is repreſented in figure 88: this ſyſtem is cauſed to turn round a vertical axis, by the action of a weight, placed in the ſcale *S*, which is ſuſpended from a line that goes ‡over the pully *R*, and is wound round the barrel *M* or *N*, the radii of which are as two to one; when the ſtring is wound round the ‖ barrel *N*, during the deſcent of the ſcale 25.25 inches, the ſyſtem makes 10 revolutions, by which it appears, that the circumference of the barrel

‡ Philoſ. Tranſ. for 1776. p. 466.

‖ Page 461.

barrel $N = 25.25$ inches, and conſequently the radius $= .401866$, and the radius of the barrel[p] $M = .803732$ parts of an inch: let the radius of the barrel M (or of the other when it is uſed) $= d$: the weight of the whole ſyſtem is 103.53 oz. avoirdupoize, let this $= w$: the diſtance of the centre of gyration[q] from the axis of motion

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[p] If Mr. Smeaton had mentioned the lengths of the barrel M and N, occupied by the line which is wound round them, eſtimated in a direction parallel to their axis, the exact circumferences and radii of the barrels might have been aſcertained from the rule in page 341. But this will not ſenſibly affect the theoretical concluſions with which the experiments are compared: the length of the barrel M occupied by 5 helical circumferences, if the line $= 25.25$ inches, will not probably exceed .4 parts of an inch, from which the true circumference of the barrel M appears to be $= 5.0493$, and its radius $= .803631$, inſtead of .803732, being the radius deduced from aſſuming 25.25 inches $=$ to 5 circular circumferences of the barrel: the times of deſcent in the experiments are not affected by this difference in the radius by ſo much as $\frac{1}{100}$ part of a ſecond, and in the other experiments, which are made on the angular velocities, the error in the times of deſcribing uniformly 20 revolutions being (in a phyſical ſenſe) evaneſcent, it is unneceſſary to apply any correction on that account.

[q] The ‡ weights and dimenſions of the ſeveral parts of the inſtrument, neceſſary for the eſtimation of the centre of gyration, are as follow, the weights being expreſſed in avoirdupoize ounces, and the diſtances in inches. K and L are two equal cylinders of lead, perforated by cylindrical cavities, in order to admit the wooden arms to which they are affixed. The weight of each is $= 48$: the external diameter $= 2.57$, the diameter of the internal cylindrical cavity $= .72$, the diſtance of the centres of the leaden cylinders $= 8.25$ from the axis: the length of the fir arm $TV = 10$; and its diameter being $= .72$, ſince the ſpecific

‡ Philoſ. Tranſ. for 1776. p. 461.

tion = 8.03383, which put = r. According to Mr. Smeaton's firſt ‡experiment, a weight

‡ Philoſ. Tranſ. for 1776, p. 463.

fic gravity of fir is about .54, when that of water is 1, the weight of the arm TV will appear to be = 1.272 oz.

The fir axle is an irregular figure, but by taking the parts of it proportionally to the reſt, its length will be = about 13, and its diameter at a medium = 1.25; this will give the weight of the axle = 4.986 oz; the weights of the ſeparate parts being added, the weight of the whole ſyſtem will appear to be = 103.53.

The diſtance between the centre of gyration, and the axis, is found by multiplying each particle of the ſyſtem into the ſquare of its diſtance from the axis, and dividing the ſum by the † weight of the whole ſyſtem: the ſquare root of the quotient, will be the diſtance required: the ſum of theſe products, muſt be found ſeparately, for each of the parts. To obtain this ſum for one of the leaden cylinders, it will be requiſite to know the weight of a cylinder of the ſame kind of lead, which would exactly fill the cylindrical cavity occupied by the wooden arm: let w repreſent this weight, R the radius of the external cylinder, r the radius of the internal cavity, W the weight of of the hollow cylinder; then will the weight required $w = \frac{Wr^2}{R^2 - r^2}$ by the properties of the cylinder, and making $W = 48$, $r = .36$, $R = 1.285$, it appears that $w = 4.08826$ = the weight of the internal cylinder of lead; if therefore the hollow cylinder, were exactly filled with lead, of equal ſpecific gravity, its weight would be = 52.08826 = $W + w$.

† Sect. VI. Prop. VIII.

The notation juſt uſed remaining, making alſo the length of the leaden cylinder = l, and the diſtance between the axis and the centre of gravity of the cylinder = g; from the rule inveſtigated in p. 223, it is eaſily inferred that the ſum of the products formed by multiplying each particle of the leaden cylinder, weighing 48 oz. into the ſquare of its diſtance $= \frac{W \times \overline{12g^2 + l^2} + 3R^2 \times \overline{W + w} - 3r^2 w}{12}$; in the preſent caſe $W = 48$, $w = 4{,}08826$, $l = 1.56$, $g = 8.25$, which being ſubſtituted, the ſum of the products required will be = 3298.103, and in both the cylinders, the ſum will be 6596.206.

weight of 8 oz. ſtretching the ſtring by being placed in the ſcale, gradually communicates motion to the ſyſtem, and deſcribes 25.25 inches in $14\frac{1}{4}$ ſeconds of time. To compare this with the theory: if w be the weight of the ſyſtem, r the diſtance of the centre of gyration from the axis, p = the moving force, d the diſtance from the axis at which it is applied, $l = 193$ inches: the time in which the deſcending weight deſcribes* the ſpace s from reſt will be $\sqrt{\frac{sd^2p + sr^2w}{lpd^2}}$ ſeconds: now in this experiment the larger barrel is uſed, of which the radius $= .803732 = d$, $p = 8$, and $d^2p = 5.1679$, $r^2w = 6682.080$, and $s = 25.25$, which quantities being ſubſtituted for their reſpective values, the time ſought $= \sqrt{\frac{5.1679 + 6682.080}{193 \times 5.1679}} \times \sqrt{25.25}$ $= 13.01$ ſeconds; by experiment the time was 14.25, which appears to be too great by 1.24 ſeconds.

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* Sect. VI. Prop. XV. Cor. 2.

From the ſame rule, making $r = 0$, $w = 0$, $W = 1.272$ being the weight of the fir arm TV, $l = 10$ being the length of the arm TV, $R = .36$, and $g = 5$, we ſhall obtain the ſum of the products of each particle in the fir arm TV, multiplied into the ſquare of its diſtance, from the axis $= 42.45$, and for both arms, the ſum will be $= 84.90$. Moreover, for the vertical axis, the ſum required = half the weight multiplied into the ſquare of the radius $= .974$, and the ſum of the products for the whole ſyſtem $= 6682.080$; and becauſe the whole weight of the ſyſtem $= 103.53$, the diſtance between the centre of gyration, and axis of motion $= \sqrt{\frac{6682.080}{103.53}} = 8.03383$.

It was alſo obſerved in the experiment, that during the deſcent of the weight 8 oz. through * 25.25 inches, ſuch an angular velocity was generated, as cauſed the ſyſtem to perform uniformly 20 revolutions in 29 ſeconds: by the theory making $c = 3.14159$, the angular velocity generated in the circumſtances of the experiment is ſuch as would cauſe the ſyſtem to deſcribe ‡ uniformly

$$\sqrt{\frac{lsp}{c^2 d^2 p + c^2 r^2 w}} = \frac{1}{3.14159} \times \sqrt{\frac{193 \times 25.25 \times 8}{5.1679 + 6682.080}} = .76858$$

parts of a revolution in a ſecond, and conſequently 20 revolutions in 26.02 ſeconds: in the experiment, the time of deſcribing 20 revolutions was 29 ſeconds, which is greater than appears from the theory by 2.98 ſeconds and parts.

By proceding in the ſame manner, the time in which the moving forces 8 oz. or 32 oz. deſcribe the different ſpaces, and the angular velocities acquired in the ſubſequent experiments, are obtained from the principles of rotation: the annexed table contains the reſults both by theory and experiment.

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* Philoſ. Tranſ. for 1776. p. 463.

‡ Sect. VI. Prop. XV. Cor. 1.

Table of Mr. Smeaton's Experiments, compared with the theory.

No.	Ounces avoirdupoize in the ſcale.	Barrel uſed *M* the larger, *N* the ſmaller.	Spaces deſcribed by the deſcending weight from reſt.	Time of the deſcent of the weight in the ſcale by theory.	Time of deſcent by experiment.	Time too great by experiment.	Time of making 20 revolutions by theory.	Time of making 20 revolutions by experiment.	Time too great by experiment.
1	8	*M*	25.25	13.01	14.25	+ 1.24	26.02	29	2.98
2	8	*N*	25.25	26.01	28.25	+ 2.24	26.01	29.25	3.24
3	8	*N*	6.31	13.00	14.25	+ 1.25	52.02	58.5	6.5
4	32	*M*	25.25	6.51	7	+ .49	13.03	14	.97
5	32	*N*	25.25	13.01	14	+ .99	13.01	14.75	1.74
6	32	*N*	6.31	6.50	7	+ .50	26.02	28.75	2.73
7	8	*M*	25.25	6.42	7	+ .58	12.84	14.75	1.91
8	8	*N*	25.25	12.83	14	+ 1.17	12.83	15	2.18
9	8	*N*	6.31	6.41	7	+ .59	25.66	30.25	4.59

The difference between theſe reſults evidently indicates a loſs of motion in all the experiments, and ſome account ſhould be given of the cauſes of this diſagreement. The weight of the ſcale *S*, which is not taken into the theoretical eſtimation, nor mentioned by Mr. Smeaton, would manifeſtly tend in the experiments to leſſen the differences, by cauſing an addition to the moving force; but then it muſt be conſidered, that three cauſes operate in a contrary way, which all together are more than ſufficient to counteract its effects: 1. the inertia of the pully *R*; 2. the friction of the pully; 3. the friction of the machine. The inertia of the pully is too ſmall to make any ſenſible alteration in the experiments, being only = to $\frac{1}{2}$ its weight multiplied into the ſquare of the radius, which muſt bear a very ſmall proportion to $wr^2 + pd^2 =$ 6682.080, to which it ſhould be added: the friction of the machine is not probably very great: when the centre ‡ of oſcillation is brought nearer to the axis in the proportion of 2 : 1 by means of counting the number of vibrations performed in 60 ſeconds, if the friction of the axis operated conſiderably, it is not probable that it ſhould act equally on the pendulum, moving with the different velocities, the weight not being altered, ſo that

‡ Philoſ. Tranſ. p. 461.

that if it were not very ſmall, the true diſtances of the centres of oſcillation could not be ſo nearly in the proportion of 2 : 1, as they appear to be from the dimenſions, &c. the greater diſtance being 8.3052, and the leſſer, when the leaden cylinders are brought nearer, being = 4.1750. If it had been mentioned at what angle the axis of motion was inclined to the horizon, when the ſyſtem performed 92 vibrations in 60 ſeconds, the effects of friction might have been in ſome degree eſtimated; for the length of ſuch a pendulum is 16.66 inches, and the length of Mr. Smeaton's pendulum when the axis is horizontal = 8.3952. Suppoſe then that the inclination of the axis to the horizon was 60°; this would have increaſed the true length in the proportion of 1 : 2, ſo as to have rendered it = 16.6104: as it appeared to be nearly of this length by experiment, when, in the author's words, the machine was converted into a kind of pendulum, it is plain, that there could have been little or no loſs of motion from friction, had the inclination juſt mentioned been 60°: if the length had come out greater by experiment, an approximation to the quantity of friction might have been obtained from the data.

It ſeems probable, however, that the

angle, at which the axis was inclined to the horizon, when it performed 92 vibrations in 60 ſeconds, was not much leſs than 60°; the friction of the pullies' axle, and of the line unwinding itſelf from the barrel, might have been the principal cauſes, which diminiſhed the effect of the moving force; but as the different reſults are affected nearly in proportion to their magnitudes, their ratios are ſcarcely altered from the truth.

When the leaden cylinders were brought nearer to the axis of motion, it was evidently meant, that the whole maſs ſhould be virtually at half its former diſtance, or in other words, if the whole were concentrated into two points ſucceſſively, ſo that the rotation ſhould not be altered from that which takes place in the conſtructed forms, the neareſt of theſe points ſhould be at a diſtance from the axis half as great as the other: but to effect this, the leaden cylinders ſhould have been moved to ſuch a diſtance from the axis, that the centre of gyration, and not the centre of oſcillation, might be at half its former diſtance; this will make ſome little difference in the eſtimation. When the centres of the leaden cylinders are moved to the diſtance of 3.92 inches from the axis, the diſtance of the centre of gyration is = 3.9596; but the diſtance ſhould have been = 4.0169, being the half of 8.0338, the original diſtance.

Although this difference will tend to leſſen the effects of friction in exp. 7, 8, and 9, yet the centre of gyration being brought much nearer to the axis, will at the ſame time operate in the contrary manner; for it is always obſerved in revolving ſyſtems, that the more near the parts are brought to the axis, the greater diminution of motion is occaſioned by friction, every thing elſe being the ſame.

‡ Philoſ. Tranſ. for 1776. p. 466, &c.

The inferences deduced by Mr. Smeaton from theſe ‡ experiments, are a very good illuſtration of the theory denominated Conſervatio virium vivarum, which, as far as regards the rotation of bodies on fixed axes, when the moving forces are conſtant, as well as other particular kinds of motion, has been demonſtrated from the Newtonian laws or axioms. As theſe inferences or principles are deduced by the author from his experiments only, he muſt certainly be conſidered to have invented them, although they had been before thought of by others: in order to compare theſe inferences with the theory, we muſt follow Mr. Smeaton's example in neglecting the inertia of the weights in the ſcales as inconſiderable: this being premiſed, let *d* denote the radius of the barrel, which is uſed in the experiment, *r* the di-

diſtance of the centre of gyration from the axis, v the velocity generated in that centre, or it may be expreſſed in the common phraſe, the velocity generated in the two bodies K and L; let $l = 193$ inches, w the weight of the ſyſtem, p the weight in the ſcale. In order to prevent miſapplication of terms, the definitions premiſed by Mr. Smeaton, may be here inſerted and expreſſed according to the notation uſed in the theory.

Phil. Tranſ. for 1776.

"By ‡ impulſe, impulſion, or impelling "force, is meant, the uniform endeavour "that one body exerts upon another, in "order to make it move:" By this moving force in the experiments, we muſt underſtand, the force of the deſcending weight p by which motion in the ſyſtem is generated: but this moving power exerts a greater force to communicate motion to the cylinders K and L (conſidered as concentrated in the centre of gyration) according as the diſtance at which it acts from the axis, or the radius of the barrel round which the line is wound, is greater, and as the diſtance of the bodies K and L from the axis becomes ſmaller; wherefore on the whole, the effect of this power to generate motion in the bodies K and L will be expreſſed by $\frac{d}{r}$, which is called by Mr. Smeaton, the intenſity with which the power

‡ Page 464.

p acts, ſo that the impelling power compounded with its intenſity will be $\frac{dp}{r}$: moreover, he defines the quantity of mechanic power by the product under the impelling force p and the ſpace through which it deſcends or by ps. By cor. 8. prop. XVI. ſect. VI. $v = \sqrt{\frac{4lsp}{w}}$, or as $4l$ is a quantity conſtantly the ſame, and the ratios only of the different quantities are inquired into, $v = \sqrt{\frac{sp}{w}}$: it follows, that $sp = v^2 w$, that is, the mechanic power is as the ſquare of the velocity into the quantity of matter moved: if therefore the quantity of mechanic power sp is unaltered, and w the weight of the ſyſtem remains the ſame, the ſquare of the velocity generated will not be altered, and the velocity of courſe the ſame, in whatever manner the times of generating this velocity may vary; as appears in Mr. Smeaton's firſt and ſecond obſervations. Moreover‡, ſince $v = \frac{2ltp}{w} \times \frac{d}{r}$, or omitting $2l$ as the relative values only are concerned, it appears that $t = \frac{vw}{p} \times \frac{r}{d}$; when v, w and p are the ſame,

1ſt & 2d Obſervation, p. 465.

‡ Sect. VI. Prop. XVI. Cor. 7.

as in the firſt and ſecond experiments and obſervations||; it follows, that the time of generating the given velocity v will be proportional to the quantity $\frac{r}{d}$: but $\frac{d}{r}$ is the direct ratio of the intenſity of the given power p, wherefore the time muſt be in theſe circumſtances in the "ſimple "inverſe ratio of the intenſity of the impulſive power;" this is affirmed in the deduction from the ſecond obſervation‡.

Phil. Tranſ. for 1776.
|| Page 465.
‡ Page 466.

Becauſe *$sp = v^2 w$, if sp the mechanic power be decreaſed in the proportion of 4 : 1, the ſquare of the velocity generated in the heavy bodies muſt be diminiſhed in the ſame proportion, w remaining conſtant, and conſequently the velocity v muſt be diminiſhed in the proportion of 2 : 1; this is the caſe in the third experiment, as explained in the third obſervation. It appears alſo from the numbers 2 and 3 in the ſixth column, that the velocity produced is as the time that a given impelling power of the ſame intenſity continues to act upon it; this is expreſſed by the §equation $t = \frac{vw}{p} \times \frac{r}{d}$: in the numbers 2 and 3 in the ſixth † column, $\frac{w}{p}$ is the ſame, and the intenſity $\frac{r}{d}$ being alſo the ſame,

3d Obſervation.
* Sect. VI. Prop. XVI. Cor. 8.
§ Sect. VI. Prop. XVI. Cor. 7.
Phil. Tranſ. for 1776.
† Table of Experim. p. 463. or p. 388. huj.

ſame, the time muſt be as the velocity acquired.

4th Obſervation, and 4th & 5th Experim. Sect. VI. Prop. XVI. Cor. 8.

Since in general $v = \sqrt{\frac{4lsp}{w}}$, if the impelling power p be quadrupled, it appears, that v will be doubled when $\frac{s}{w}$ is given: in the experiment referred to in this obſervation, and in the 5th experiment, the velocity of the bodies is increaſed in that proportion, when compared with the velocity generated in the firſt experiment.

It is ſaid in this obſervation, that the velocity acquired is ſimply as the impelling power compounded with the time of its action: "for a quadruple impulſion "acting for 7″, inſtead of 14″, generates a "double velocity:" but we muſt here underſtand the impelling power to mean that power compounded with its intenſity, which (the ratio of the time in which it acts being added,) is proportional to the velocity generated. In experiment 1 and 4, the intenſity is the ſame, $\frac{d}{r}$ being given in both caſes; therefore in theſe inſtances, the velocity generated in the heavy bodies will be ſimply as the impelling power. But in the 1ſt and 7th experiments, the impelling power being the ſame, the intenſities meaſured by $\frac{d}{r}$ are as 1 : 2, and the times of action being as 2 : 1, theſe proportions compounded give the ratio of equality in the * velocities generated, as they appear in the experiments: if the impelling powers or weights in the ſcale had been different, the proportion of the two weights muſt have been compounded with the others.

Since $v = \frac{p}{w} \times \frac{td}{r}$, $\frac{d}{r}$ being the intenſity of the force p, by doubling $\frac{d}{r}$ and diminiſhing t one half, $\frac{td}{r}$ is not altered, and conſequently $\frac{p}{w}$ being conſtant, the velocity $= \frac{ptd}{wr}$ muſt be the ſame as before. This is what is expreſſed in the ſixth obſervation ‡; where it is ſaid, that "an impulſive power of double the intenſity acting for half the time, produces "the ſame effect in generating motion, "as an impulſive power of half the intenſity acting for the whole time."

In the 7th obſervation*, Mr. Smeaton infers it to be "an univerſal law of nature, that the mechanic powers to be

* P. 395. & Sect. VI. Prop. XVI. Cor. 7.

Phil. Tranſ. for 1776. 5th & 6th Obſervation, and Experiment, p. 467.

‡ Page 468.

* Pag. 469.

"expended" (meaſured by the weights and the ſpaces through which they deſcend jointly) "are as the ſquares of the "velocities to be generated;" and "that "the velocities generated are as the im- "pelling power compounded with or "multiplied by the time of its action "and vice verſâ." Here alſo by the impelling power muſt be underſtood, the moving force or weight compounded with its intenſity; which quantity (the ratio of the time in which it acts being added) has appeared in the experiments, as well as from the theory, to be, as Mr. Smeaton affirms, proportional to the velocity generated in the revolving bodies.

The axle of Mr. Smeaton's inſtrument makes no ſenſible alteration in the time wherein the deſcending weight deſcribes a given ſpace, or in the angular velocity generated; ſo that, whether it be taken into account or neglected, will be wholly immaterial; nor will there be a perceptible difference in the reſult, whether the axle be made of the lighteſt or heavieſt kind of wood. To exemplify this, let the firſt experiment of Mr. Smeaton's be aſſumed as an example; and let the axis be neglected: the time of deſcribing* 25.25 inches will be 13.0103 ſeconds; if the axis be made of fir wood, the time will be 13.0113: laſtly, if the axis ſhould be of mahogany, the ſpecific gravity of which is about 1.1, when that of water is = 1, the time will appear to be = 13.0122.

Here we obſerve that it makes a difference of no more than a five hundredth part of a ſecond, whether the axle be wholly neglected, or be taken into account on a ſuppoſition that its ſpecific gravity is the ſame as that of mahogany. Mr. Smeaton does not mention of what ſubſtance his axle was formed; he deſcribes the arms to have been fir; in the computation‡ the axle was ſuppoſed to be

* Table of Experim. p. 388.

‡ Page 385.

of that ſubſtance, but it appears from this note, that the ſpecific gravity of the axle is wholly immaterial, and that the numbers in the table p. 388. will not at all be altered, whatever be the ſpecific gravity of the wood of which the axle was formed.

The numbers inſerted in pages 391 and 392, relating to the diſtances of the centre of gyration, will be altered by the different ſpecific gravities of the wooden axle, but they will be altered proportionally, ſo as to remain in a phyſical ſenſe in the ſame ratio as is there expreſſed: the reaſoning therefore to which theſe numbers are applied, will not be effected by any alteration in the ſpecific gravity of the axle.

SECT. X.

CONCERNING THE ROTATION OF BODIES IN FREE SPACE, AND ON THE CENTRE OF SPONTANEOUS ROTATION.

THE motion of bodies in free ſpace, that is, of bodies at liberty to move freely by the action of any force impreſſed, being unoppoſed by a fixed axis or other obſtacle, is immediately deduced from the principles of rotation about fixed axes, which have been demonſtrated in Sect. VI.

The fixed axis round which a body revolves, is preſſed by the impelling force while it generates rotatory motion; but the axis being (by the hypotheſis) immoveable, reacts equally againſt that preſſure, and *when it paſſes through the centre of gravity, the force of preſſure urging the axis to motion is ſuch, as, if un-

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* In the propoſitions throughout this ſection relating to the motion of bodies in free ſpace, the plane of rotation is ſuppoſed to be unaltered in reſpect of the revolving body, conſequently in the examples of bodies revolving on fixed axes, aſſumed to illuſtrate the ſubject, the preſſure againſt the axis muſt be conſidered as equal in every part, ſo that if the axis be unoppoſed, it may move by the action of the impelling force always parallel to itſelf.

unoppoſed, would cauſe each particle of the ſyſtem to move with the ſame velocity, and in the direction in which the force acts. If then the force which preſſes againſt a fixed axis, on which a body revolves in given circumſtances be aſcertained, the motion of the body in free ſpace, when the axis is removed, will be known. For the motion of the body in free ſpace, will conſiſt of the rotatory motion round the axis, paſſing through the centre of gravity conſidered as fixed, which is determined by the propoſitions in Sect. VI. compounded with the motion of the centre of gravity cauſed by the force now free to impel that centre, the fixed axis which paſſed through it being removed.

By the centre of gravity in any body or ſyſtem of bodies, is uſually underſtood to be meant, a point in which the gravitation of each individual particle may be ſuppoſed concentrated; becauſe if the gravity of each particle in reality acted on that point, the ſame motion would be produced in the ſyſtem, as when the gravity of each particle acts at its reſpective diſtance from the centre of gravity, the direction of theſe forces being always parallel to each other. But the point which anſwers this deſcription, being determined in any ſyſtem by the known geometrical rules, many properties relating to the quieſcence as well as the motion of bodies, are demonſtrated to belong to it wholly independent of gravity, and which would be equally efficient if gravity exiſted not. Thus, if each particle of a body without gravity, moves in free ſpace with an uniform velocity, and an immoveable obſtacle is oppoſed to that centre, the whole motion of the ſyſtem will be deſtroyed; whereas if the obſtacle be oppoſed to any other point, the ſyſtem will continue to move although each particle will ſuffer alteration in the velocity and direction of its motion.

Moreover, if a ſyſtem revolves in free ſpace round any axis which paſſes through the centre of gravity, that centre will not be affected by the action of the parts of the ſyſtem on each other; if therefore the centre of gravity be quieſcent when the rotation commences, it will continue quieſcent during the rotation of the ſyſtem round it, nor will it change its place until ſome external force is impreſſed upon it.

If by any force acting for a portion of time, either ſmall or great, motion ſhould be communicated to this centre, after the force ceaſes to impel it, the centre of gravity will move on uniformly in a right line with the velocity acquired, having power to alter neither the velocity nor determination of its motion*.

Theſe properties are evidently independent of gravity and other ſimilar powers, which ſhews that the point uſually denominated the centre of gravity, might with equal propriety be called the centre of inertia, when applied to the motion of bodies in free ſpace, eſpecially ſince gravity or other motive power may be cauſed to act partially on a ſyſtem, and it is often neceſſary to conſider a power ſo acting in theory in order to demonſtrate ſome properties of motion, in which caſe the centres of gravity and inertia do not coincide, this coincidence happening when each particle in a ſyſtem is impelled by gravity or other ſimilar power, which acts in right lines parallel to each other, and which is proportional to the quantity of matter moved, and in other particular caſes.

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* The principles of centripetal and centrifugal forces, are demonſtrated on a ſuppoſition, that the centre of gravity proceeds in a right line until it is compelled to change its direction by the agency of ſome external force. Let *G* be the centre of gravity of a ſyſtem impelled in the direction *Gg*; then will *G* continue to move in the direction *Gg*, until ſome external force ſhall act on it: if any force of this kind always tends towards a centre *C*, and is ſufficient to retain the centre of gravity *G* in the circular arc *GB*, the quantity of this centripetal force will be meaſured by the line *Bg* or the verſed ſine of the arc *GB*, this arc being in its evaneſcent ſtate: and the ſame line *Bg* will meaſure the force by which *G* endeavours to recede from the centre, and as ſoon as the central force ceaſes to act, will proceed in the direction of a tangent to the arc which it is deſcribing when the central force ceaſes. converſely from the properties of centripetal and centrifugal forces aſſumed as true, the rectilinear motion of the centre of gravity might be inferred. Fig. XC.

Fig. XCI. To illuſtrate theſe remarks by an example, let A, B, C, D, E, F be ſmall bodies or material points without gravity, ſituated in the right line AF, and let them be united by ſome perfectly rigid ſubſtance without weight or gravity. If AG be made $= \frac{A \times 0 + B \times BA + C \times CA, \&c.}{A + B + C, \&c.}$, then will G be the point denominated the centre of gravity of the bodies A, B, C, &c. or according to the reaſoning juſt alledged, ſince theſe bodies are by the ſuppoſition void of gravity, G may be termed the centre of inertia. Moreover, let forces of any kind, repreſented in quantity by the letters a, b, c, d, e, f, act on the ſyſtem at the points a, b, c, &c. reſpectively in any given direction, for example in a direction perpendicular to AF: if Ag be made $= \frac{a \times Aa + b \times Ab + c \times Ac, \&c.}{a + b + c, \&c.}$, then will g be the centre of gravity of the ſyſtem, properly ſo called, and the forces a, b, c, &c. will have the ſame effect to communicate motion to the ſyſtem, as if the ſum of the forces $a+b+c$, &c. acted altogether in the point g.

Here we obſerve, that the points G and g, which have been denominated the centres of inertia and gravity, may be diſtant from each other by any ſpace within the limits of the ſyſtem, but will coincide when the forces by which each particle of a body is impelled, act in parallel lines, and are proportional to the quantities of matter contained in the particles moved.

Suppoſe the ſum of the forces $a+b+c$, &c. to act at g, and to be repreſented in quantity by g: concerning the application of theſe principles, it is alſo to be remarked, that if the force g, &c. be of the ſame kind as gravity, the elaſtic force or preſſure of a fluid without inertia, the place of the point G will not be altered by the application of the force g. But if g be a force which poſſeſſes inertia, G will begin to move from its place the inſtant g is applied to act on the ſyſtem, and will approach nearer to g, the greater proportion the inertia of g bears to that of $A+B+C$, &c. But as taking into conſideration the motion of the point G during the action of the force applied at g, would render the enſuing propoſitions too complex: it will be expedient to conſider the inertia of the force applied to move a ſyſtem in free ſpace, as evaneſcent, as far as regards any motion of the point *G* which may be occaſioned by it.

A diſtinction between the centres of inertia and gravity was inſerted only for the purpoſe of explaining the ſubject in queſtion; but there is no neceſſity to introduce the uſe of a new term, as the centre of gravity of a ſyſtem ſuch as $A + B + C$, &c. without weight, will be always underſtood as determining the poſition of that point in the ſyſtem, and not as implying the exiſtence of gravity.

Fig. XCII. Let *SIK* repreſent a plane, into which the matter contained in a ſyſtem is projected: let *G* be the centre of gravity through which a fixed axis of motion paſſes perpendicular to the plane *SIK*; let motion be communicated to the ſyſtem by means of a line *DMEDp*, wound round the circle *EDM*, of which the centre is *G*, and ſtretched by a weight or force of any kind *p*; then the axis ſuſtains the weight of the ſyſtem *SIK* added to the tenſion of the ſtring *Dp*, or if the ſyſtem be conſidered as without weight, the preſſure againſt the axis will be the tenſion of the ſtring *Dp* only. Now ſuppoſe the axis to be at liberty to move in free ſpace (always however being parallel to its firſt poſition) then the weight *p*, acting by ſtretching the ſtring *D p*, will urge the centre of gravity *G* by a force equal to the ſtring's tenſion, while the ſyſtem revolves round *G* by the action of the ſame force. In order therefore to aſcertain the motion of the ſyſtem in free ſpace, it will be neceſſary to determine the tenſion of the ſtring *D p* while the ſyſtem revolves round the fixed axis which paſſes through *G*.

I.

Fig. XCII.

Let *SIK* repreſent a ſyſtem which is moveable round a fixed axis paſſing through its centre of gravity *G*: with the centre *G* and any diſtance *G D*, let a circle *D E M* be deſcribed, and let motion be communicated by a weight or force *p*, ſtretching a line which is wound round the circumference of the circle *M E D*; it is required to aſſign the tenſion of the ſtring *D p*, while the ſyſtem is accelerated in its angular motion round *G*.

Let *R* be the centre of gyration of the ſyſtem, and let $GR = r$, $GD = d$, the weight or inertia of the ſyſtem $= w$; then the acceleration of the ſyſtem will be the ſame as if the whole maſs being removed, that part of it which is expreſſed* by the fraction $\frac{wr^2}{d^2}$ were uniformly accumulated in the circumference *M E D*: moreover, ſince the force which accelerates the circumference is that part of the acceleration of gravity which is expreſſed by the fraction

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* Sect. VI. Prop. VIII. Cor. 2.

† Sect. VI. Prop. XV. $\frac{pd^2}{pd^2+wr^2}$, † multiplying this into $\frac{wr^2}{d^2}$, we have the
‡ Sect. I. Prop. IX. tenſion of the ‡ ſtring $Dp = \frac{pwr^2}{pd^2+wr^2}$.

Cor. 1. The quantities p and w are in this propoſition underſtood, as uſual, to repreſent both the weight and inertia of the bodies reſpectively. If it ſhould be preferred to repreſent the inertia and weight by different quantities, the ſolution will in no reſpect be altered: thus, let w be the abſolute weight or gravity of the ſyſtem, q its inertia, p the abſolute force which acts on the point D, i its inertia, then will the force which accelerates the circumference of the circle DEM be $\frac{pd^2}{id^2+qr^2}$, which being multiplied into the equivalent maſs $\frac{qr^2}{d^2}$ will give the ſtring's tenſion $= \frac{pqr^2}{id^2+qr^2}$.

Cor. 2. It appears that in whatever manner the weight of the ſyſtem w may vary, the tenſion of the ſtring will not be altered.

Cor. 3. If the inertia of the force which ſtretches the ſtring, that is, if i be evaneſcent, the tenſion of the ſtring will become $= p$; and this will be the caſe whatever be the magnitude of the force p.

Cor. 4. When r is infinite, although p, q, i and d remain finite, the tenſion in this caſe alſo becomes $= p$.

Cor. 5. When i the inertia of the moving force is evaneſcent, the preſſure on the axis, which is equal to the ſtring's tenſion, will be $= p$. The ſyſtem therefore being without weight, and the axis paſſing through G being ſuſ-
Fig. XCIII. tained by a line GA *which goes over a fixed pully, if the line GA be ſtretched by a weight p, it will exactly counterbalance the preſſure on the axis occaſioned by the tenſion of the ſtring Dp as p deſcends, ſo that G will remain quieſcent.

II.

Fig. XCIV. Every thing elſe remaining, let the axis which paſſes through the centre of gravity G, be moveable in free ſpace; it is required to determine the point round which the ſolid begins to revolve in free ſpace at the firſt inſtant of motion.

Let G be the centre of gravity of the ſyſtem, being coincident with the centre of the circle DME, round which the line $DMEDp$ is wound; a force or weight p by ſtretching the line Dp, acts on the ſyſtem in the direction of a tangent Dp to the circle at the point D, which is perpendicular to GD.

Produce DG indefinitely: The points G, D, &c. in the line OGD muſt begin to move in the direction of the impelling force; that is, in a direction which is perpendicular to OD: and becauſe the determination of motion once impreſſed on the centre of gravity, is not altered, except by the impulſe of external force acting in ſome † other direction, it follows, that the direction in which the point G proceeds from the very beginning of its motion, will be perpendicular to the line OD.

† Page 403.

Draw Gg perpendicular to OD, being equal to the ſpace through which the tenſion of the ſtring applied to act on the centre of *gravity would impel it in any particle of time t: through g draw od parallel to OD; make dge equal to the angle deſcribed about G, conſidered as fixed during the time t; through e and g draw egO interſecting OGD in the point O; about this point the ſyſtem will begin to revolve in free ſpace.

* Sect. X. Prop. I. Cor. 3.

Make $go = GO$ and join Oo: then ſince by conſtruction Oo is equal to Gg, Oo is the ſpace through which the force impreſſed on the centre of gravity carries the point O, in the direction Oo parallel to Gg; and oO is the ſpace through which the rotation of the ſyſtem carries the point O in a contrary direction in the ſame time t; and as this is applicable every inſtant during the motion of the point G through the evaneſcent ſpace Gg, it follows, that while the other points of the ſyſtem are changing their places, the point O will be quieſcent, or in other words, the ſyſtem will begin to move round the point O, as determined by this conſtruction.

III.

The conſtruction remaining, if gravity be ſuppoſed to act on the ſyſtem, and a fixed horizontal axis be cauſed to Fig. XCIV.

paſs through the point *O* perpendicular to the plane *SIK*, *D* will be the centre of oſcillation.

When the ſyſtem is impelled by the force p in free ſpace, let t be a very ſmall portion of time in which motion is produced in the ſyſtem: let Q be the principal centre of gyration, that is, the centre of gyration when the ſyſtem revolves in the plane *SIK*, round an axis which paſſes through the ‖centre of gravity; and let $GQ = r$, w = the weight of the ſyſtem, $l = 193$ inches, $GD = d$: then ſuppoſing a fixed axis to paſs through *G*, the ſpace through*which the weight p deſcends in the time t or the evaneſcent arc de will $= \frac{plt^2d^2}{d^2p + wr^2}$, or p being very ſmall, that is, pd^2 being inconſiderable when compared with wr^2, $de = \frac{plt^2d^2}{wr^2}$: Secondly, When the centre of gravity *G* is at liberty to move freely in the direction of the impelling force p, being unoppoſed by a fixed axis, the ſpace $Gg = Dd$ deſcribed in the † time t will be $= \frac{pt^2l}{w}$; wherefore $De = \frac{\overline{r^2 + d^2} \times pt^2l}{wr^2}$: and becauſe of the ſimilar triangles *DOe*, *GgO*, we have $De, Gg :: DO : GO$, or as $DO : GO :: \frac{\overline{r^2 + d^2} \times pt^2l}{wr^2} : \frac{plt^2}{w} :: r^2 + d^2 : r^2$; wherefore $DO - GO : GO :: d^2 : r^2$, that is, ſince $d = GD = DO - GO$, and $r = GQ$, we have $DG^2 : GQ^2 :: DG : GO$, and $GO = \frac{\overline{GQ}^2}{DG}$; but GQ being the diſtance of the principal centre of gyration from the ‡ centre of gravity, whenever $GD \times GO = GQ^2$, if *D* be made the centre of ſuſpenſion, *O* will be the centre of oſcillation, and vice verſâ: from which it follows in the preſent caſe, that *O* is the centre of oſcillation of the ſyſtem, when it vibrates freely round an horizontal axis paſſing through *D* perpendicular to the plane *SIK*.

The demonſtration here given, might be applied to the rotation of bodies in free ſpace, which is effected by percuſſion, ſince it has been ſhewn, that motion communicated

‖ Sect. VI. Prop. VIII.

* Sect. VI. Prop. XV. Cor. 2.

† Sect. III. Prop. IV. Cor. 5.

‡ Sect. VI. Prop. VIII. Cor. 3.

nicated in this manner, is the effect of ‡continual acceleration: but for the ſake of explaining the propoſition more diſtinctly, the following demonſtration may be ſubjoined, a few conſiderations relating to the ſubject being premiſed.

‡ Page 8. & Sect. IX. p. 377.

Fig. XCV.

Let *ABCD*, &c. be a ſyſtem of bodies conſiſting of particles or material inert points without gravity, being connected together by ſome perfectly rigid ſubſtance without weight or inertia; and ſuppoſe theſe points to be †projected into the plane *AODBK* by right lines perpendicular to it: let the ſyſtem be moveable round a fixed axis paſſing through *O*, and perpendicular to the plane *AODBK*. Through *O* and the centre of gravity *G* draw the line *OGE*, and let a body impinge on the ſyſtem in the direction *LE* perpendicular to *OGE*, and in the plane *AODBK*. This impulſe will urge the ſyſtem round the axis which paſſes through *O*, each body beginning to deſcribe a circular arc of which the centre is the point *O*, and the radius the diſtance of the body from that point reſpectively. But each body by its inertia will reſiſt the communication of motion in a direction contrary to that in which it is impelled; thus the body *A* beginning to move in the direction *Aa*, which is perpendicular to *OA*, will reſiſt the communication of motion in the direction *aA*: in the ſame manner the point *B* will reſiſt the communication of motion in the direction *bB* perpendicular to *BO*, and ſo on of the reſt. It is plain, that the effect of the impact at *E* on the point *O* will depend on the reſiſtance of inertia which is oppoſed by the particles of the ſyſtem to the communication of motion. If, for example, the greater part of this reſiſtance acts between *E* and *a*, then the impulſe at *E* will be applied between the point O and reſiſtance of inertia, which will ſerve as a fulcrum for the lever to act againſt *O*; the point *O* therefore will be propelled by the impact in the direction of the ſtroke: but if the reſiſtance of inertia acts chiefly between *O* and *E*, the impulſes at *E* and *O* will be on oppoſite ſides of the reſiſtance of inertia, which ſerving as a fulcrum for the lever to act againſt, will cauſe the point *O* to be impelled by the impact in a direction contrary to that of the ſtroke, and conſequently if the inertia is diſpoſed ſo as to act on the ſyſtem to impel the point *O* equally in oppoſite directions, the axis paſſing through *O* will be not be urged either in one direction or the other, while the ſyſtem begins to revolve round it. In this caſe therefore the fixed axis will not at all affect the motion of the ſyſtem,

† Sect. VI. Prop. I. Cor. 5.

tem, which when ſtruck in the point E will begin to revolve ſpontaneouſly about the point O in free ſpace, when the fixed axis paſſing through it is removed. The point O anſwering to this deſcription, is called the centre of ſpontaneous rotation*.

‡ Bernoulli, Vol. IV. p. 265.

IV.

Fig. XCV.

Let E be the point in the ſyſtem $ABCD$, which is ſtruck by a body impinging in the direction LE perpendicular to OGE, and let O be the centre of ſpontaneous rotation; then if O is the point round which the ſyſtem revolves on a fixed axis perpendicular to the plane $ADGB$, E will be the centre of percuſſion.

Through the points A, B, C and D draw Aa, Bb, Cc and Dd perpendicular to AO, BO, CO and DO reſpectively; and AK, BH, CI and DF perpendicular to OGE, and let O be any point round which the ſyſtem is moveable by the force of the impact at E: at the inſtant of the impact, the body A will be impelled in the direction Aa, and conſequently will oppoſe a reſiſtance of inertia in the oppoſite direction aA. The force of the impact at E to impel the point O will depend on the inertia of each particle, that of A for example, acting on it by the lever, of which the arms are Ea, EO, the force of the impact at E being the fulcrum: and although the fulcrum at E be not immoveable, and will oppoſe different degrees of force, according to the different velocities and quantities of matter in the impinging body, yet in a ſingle impact, the force at the fulcrum E will be the ſame in reſpect of all the particles A, B, C and D; and therefore the relative values of the forces by which the inertia of the different particles A, B, C and D, &c. impels the point O will be obtained from conſidering E as an immoveable fulcrum.

This being conſidered, the inertia of the particle A which it oppoſes to the communication of motion at E is

$\frac{A\times AO^2}{OE^2}$*; and conſequently equal to the force of inertia $\frac{A\times OA}{OE}$ acting at A: this inertia acting on the line Oa in the point a, and in the direction aA may be repreſented by that line Aa in quantity and direction; but this force acting obliquely muſt be reſolved into two KA, perpendicular to Oa and Ka coincident with it, of which KA only is efficient to oppoſe the angular motion of the ſyſtem: this will be to the whole inertia as AK to Aa, or becauſe the triangles AKa and AOK are ſimilar, as OK to OA; we have therefore the inertia oppoſed at a to the communication of angular motion by the particle $A = \frac{A\times OA}{OE}\times\frac{OK}{OA} = \frac{A\times OK}{OE}$, acting at a in a direction perpendicular to Oa. Now the impact being made at E, the inertia oppoſed by A at the point a, in a direction perpendicular to Oa, acting againſt E as a fulcrum, will impel the point O in a direction contrary to that of the ſtroke; but the effect of any force applied at a to urge O round the point E will be that force $\times\frac{Ea}{OE}$, that is, the force of A's inertia which impels O will be $\frac{A\times OK\times Ea}{OE^2}$: in the ſame manner, the force to impel O ariſing from the inertia of C will be $\frac{A\times OI\times Ec}{OE^2}$ acting in the ſame direction; the forces which ariſe from the inertia of B and D will impel O in a contrary direction, and will be expreſſed by $\frac{B\times OH\times Eb}{OE^2}$ and $\frac{D\times OF\times Ed}{OE^2}$ reſpectively: but becauſe O is the centre of ſpontaneous rotation by the propoſition, it is not affected by the impulſes communicated to the ſyſtem, being quieſcent while the other parts of the ſyſtem begin to move with velocities which are proportional to their diſtances from it. The forces therefore which act on O in contrary directions muſt be equal, which will give $\frac{A\times OK\times Ea + C\times OI\times Ec}{OE^2} = \frac{B\times OH\times Eb + D\times OF\times Ed}{OE^2}$: or becauſe $Ea = Oa -$

* Sect. VI. Prop. II.

$OE = \frac{OA^2}{OK} - OE$, $Ec = Oc - OE = \frac{OC^2}{OI} - OE$, $Eb = OE - Ob = OE - \frac{OB^2}{OH}$, and $Ed = OE - Od = OE - \frac{OD^2}{OF}$, by ſubſtituting theſe values and mutiplying by OE^2, we have $A \times OA^2 - A \times OK \times OE + C \times OC^2 - C \times OI \times OE = B \times OH \times OE - B \times BO^2 + D \times OF \times OE - D \times OD^2$, from which we obtain $OE = \frac{A \times OA^2 + B \times OB^2 + C \times OC^2 + D \times OD^2}{A \times OK + B \times OH + C \times OI + D \times OF}$, or $OE = \frac{A \times OA^2 + B \times OB^2 + C \times OC^2 + D \times OD^2}{\overline{A + B + C + D} \times OG}$, which is the diſtance of the centre of percuſſion from the axis of

* Sect. VI. Prop. VII. motion, *when that axis paſſes through O.

Cor. 1. The ſame demonſtration preciſely is applied to any indefinite number of material points which conſtitute natural bodies.

Cor. 2. If W be the maſs contained in the ſyſtem, and R be the centre of gyration when it revolves round O; a given body impinging perpendicularly againſt E, the ſyſtem exiſting in free ſpace, will generate the ſame motion in that point, as if the maſs $\frac{W \times OR^2}{OE^2}$ were concentrated in E, the matter of the ſyſtem being removed: for the ſame reaſon, if the entire ſyſtem moves in the direction EL, which is always perpendicular to OE, and an immoveable obſtacle at E be ſtruck by it, the force of the impact will be the ſame as if the obſtacle were ſtruck by a

‡ Sect. VI. Prop. VIII. body $= \frac{W \times OR^2}{OE^2} = \frac{W \times OG}{OE}$ ‡ concentrated into the point which ſtrikes on the obſtacle, with a velocity equal to that with which the ſyſtem moved.

Fig. XCVI. Cor. 3. Let SIK be a ſyſtem exiſting in free ſpace, G being the centre of gravity, E the point on which a body impinges in the direction LE perpendicular to GE, and let O be the centre of ſpontaneous rotation; if the impact be made on the point F or any other point in the line FE inſtead of on E, the centre of ſpontaneous rotation O will not be altered if the direction of the impact remains the ſame; becauſe it is indifferent as to the particles which compoſe the ſyſtem, whether the impulſe is communicated to them from one point of the given line *EF*, or from any other: moreover, when the ſyſtem revolves about an axis which paſſes through *O*, any point in the line *EF* will be a centre of percuſſion.

Cor. 4. Since when an impulſe of any magnitude is impreſſed on *E*, the centre of ſpontaneous rotation *O* is not affected by it, it follows, that if during the motion of the ſyſtem in the direction *HI* parallel to *EF*, the point *E* ſhould impinge againſt an immoveable obſtacle, the motion of the point *O* will not be affected by it: the point *O* will in this caſe proceed after the impact with a velocity equal to that with which it before moved.

Cor. 5. If any point *F* in the line *EF* impinges in the direction *EL* againſt an immoveable obſtacle, the motion of the point O will not be affected by it, provided the ſurface of the impinging body at *F* be perpendicular to the direction in which the point *F* ſtrikes the obſtacle, and both be ſo ſmooth and hard that there ſhall be no adheſion or friction between them: for the point O endeavouring to proceed with a velocity equal to that which it had before the impact, when the point *E* is ſtopped will cauſe the ſyſtem to revolve round *E* for a ſmall particle of time, and conſequently the ſurface of the body at *F* muſt ſlide over the fixed obſtacle for a ſmall ſpace: it is only on this ſuppoſition, that the force of the impulſe on the obſtacle at *F* is the ſame as that which it would receive if an equivalent maſs equal to that of the ſyſtem multiplied into the fraction $\frac{OG}{OE}$ impinged on it with the ſame velocity. If the ſurfaces at *F* adhere, it is manifeſt that the point O inſtead of proceeding in a direction parallel to *EF*, will be deflected into a direction perpendicular to *OF*, and will conſequently loſe part of its motion eſtimated in the direction *EF* or *HI*.

Cor. 6. Since when a ſyſtem is ſtruck in free ſpace, the point denominated the centre of ſpontaneous rotation is quieſcent while the other parts of the ſyſtem begin to revolve, it is plain that the abſolute velocity of the point ſtruck, that of the centre of gravity, and the angular velocity generated will be the ſame as if a fixed axis paſſed through the centre of ſpontaneous rotation, we may therefore refer to what has been before demonſtrated in ſect. VI. in order to determine theſe quantities.

Poſtulate for the enſuing propoſitions. It is required that in the bodies or ſyſtem of bodies, which revolve round an axis perpendicular to any conſtant plane of motion, the ſum of the products which are formed by multiplying each particle into the ſquare of its diſtance from the axis paſſing through the centre of gravity may be determined. If the figure be regular, this may be effected by geometrical rules, if irregular, it may be obtained from experiment by the rules given in page 227.

V.

Fig. XCVI. Let *SIK* repreſent a body exiſting in free ſpace, and let a body impinge on it in any given point *F*, and in any direction *LF*: having the maſs contained in the body ſtruck, the velocity of the impact, the perpendicular diſtance of its direction *GE* from the centre of gravity, and the quantity of matter in the ſtriking body; it is required to aſſign the velocity of the centre of gravity, and the angular velocity generated by the impact, ſuppoſing the ſyſtem to be moveable in the plane *SIK* only.

Let *G* be the centre of gravity of the ſyſtem; produce *LF* indefinitely in the direction *LFM*, and through *G* draw *OGK* perpendicular to *LM*. Suppoſe the ſyſtem to revolve in the plane *SIK*, on a fixed axis which paſſes through *E*, and let *O* be the centre of percuſſion; this point *O* will alſo be the centre of ſpontaneous rotation, about which the ſyſtem begins to revolve in free ſpace by the force * of the impact at *F*. The point *O* being therefore immoveable at the firſt inſtant of the ſyſtem's revolution, the angular velocity generated and the initial velocity of the centre of gravity will be the ſame as when a fixed axis paſſes through *O*. Let *W* be the quantity of matter in the ſyſtem, *A* that of the ſtriking body, and *V* the velocity of impact: alſo let *R* be the centre of gyration, when the ſyſtem revolves round *O*; then the velocity generated in *E* by a given impact at *E*, will be

* Sect. X. Prop. IV. Cor. 3.

be the ſame as if the whole maſs of the ſyſtem were removed, and the ‡equivalent maſs $\frac{W \times OR^2}{OE^2}$ were concentrated in E, and the bodies being nonelaſtic, the laws of colliſion give us the following proportion, as $A + \frac{W \times OR^2}{OE^2} : A :: V$ to the velocity communicated, if A impinged on the point E; but here as F is at a greater diſtance from the centre of motion O than E, the inertia of A will be greater in a duplicate †proportion, and will therefore be $= \frac{A \times FO^2}{OE^2}$, which maſs being concentrated with the reſt at E, will be equivalent to the inertia of A; this being the caſe, the laws of colliſion will give the proportion: as $\frac{A \times FO^2 + W \times OR^2}{OE^2} : A :: V :$ to the velocity communicated to E, which will be $= \frac{AV \times OE^2}{W \times OR^2 + A \times OF^2}$: and conſequently this being diminiſhed in the proportion of OE to OG will become $\frac{AV \times OE \times OG}{W \times OR^2 + A \times OF^2} =$ the initial velocity of the centre of gravity. Moreover, for the angular velocity, making $3.14159 = C$, we ſhall have the circumference of the circle which would be uniformly deſcribed by the point E round O, if O were fixed, $= 2C \times OE$; which will give this proportion: as $\frac{VA \times OE^2}{W \times OR^2 + A \times OF^2} : 2C \times OE$, ſo is 1 ſecond to the time of one revolution in ſeconds $= \frac{2C \times \overline{W \times OR^2 + A \times OF^2}}{VA \times OE}$, (1 ſecond being the ſtandard time to which the velocities are referred;) and conſequently the number of revolutions or parts of a revolution is a ſecond, or the angular velocity round the point O required $= \frac{AV \times OE}{2C \times \overline{W \times OR^2 + A \times OF^2}}$.

Cor. 1. If the inertia of the ſtriking body be evaneſcent, the velocity communicated to the point E will

‡ Sect. VI. Prop. VIII. Cor. 2. & 4.

† Sect. VI. Prop. II.

* Sect. VI. Prop. VIII. Cor. 1.

become $= \frac{AV \times OE^2}{W \times OR^2}$, or becauſe *$OR^2 = OE \times OG$, the initial velocity of E will $= \frac{VA \times OE}{W \times OG}$.

Cor. 2. For the ſame reaſon the velocity of the centre of gravity will become $= \frac{VA \times OE \times OG}{W \times OR^2} = \frac{VA \times OE \times OG}{W \times OE \times OG} = \frac{VA}{W}$, being the ſame as would be generated in the centre of gravity G if the body A impinged directly on it with the velocity V.

Cor. 3. The inertia of P being evaneſcent as in the laſt corollaries, we have alſo the angular velocity generated in the ſyſtem $= \frac{AV}{2CW \times OG}$.

‡ Sect. VI. Prop. VIII. Cor. 3.

Cor. 4. Let Q be the centre of gyration of the ſyſtem when it revolves round the centre of gravity; then $GQ^2 = GO \times GE$ ‡; wherefore the angular velocity generated is equal to $\frac{VA \times GE}{2CW \times GQ^2}$. From this and the laſt cor. it follows, that every thing elſe being the ſame, the angular velocity generated will be directly as the diſtance of the impact from the centre of gravity GE, and inverſely as OG, GQ being a conſtant quantity in a given ſyſtem, revolving in the ſame plane.

VI.

The conditions of the laſt propoſition remaining, let it be required to aſſign the motion of the centre of gravity in free ſpace, and the angular velocity of the ſyſtem round it.

Fig. XCVII.

O being the centre of ſpontaneous rotation, each particle in the ſyſtem after the impulſe begins to move in a direction perpendicular to the line which joins the particle and the point O reſpectively. If a fixed axis paſſed through that point, each particle would continue to deſcribe uniformly with the velocity acquired a circular arc, of which the radius is the diſtance between the moving particle and the axis of motion reſpectively. But the point *G* not being confined by the action of a fixed axis, will go on in the direction of its firſt † impulſe perpendicular to *OE*, having deſcribed the right line *Gg* with the velocity $\frac{AV\times OE\times OG}{W\times OR^2+A\times OF^2}$ *, and if no rotatory motion were communicated to the ſyſtem, the line *OGE* would have moved into the poſition *ogE* parallel to *OGE*; but the angular motion deſcribed about *O* during the ſame time is = the angle *EOG* = *Ogo* the cotemporary angular velocity round the centre of gravity. The motion therefore of the ſyſtem in free ſpace will be compounded of the uniform rectilinear velocity generated in the centre of gravity, in the direction *Gg* perpendicular to *OE*, and the angular motion generated round the centre of gravity, which is equal to that round the point *O*, or = $\frac{VA\times OE}{2C\times\overline{W\times OR^2+A\times OF^2}}$.

Cor. 1. When the inertia of *A* is very ſmall, the angular velocity round the centre of gravity will become $\frac{VA\times OE}{2C\times W\times OR^2}$, or becauſe ‡ $OR^2 = OG\times OE$, the angular velocity $= \frac{VA\times OE}{2C\times W\times OG\times OE} = \frac{VA}{2C\times OG\times W}$ $= \frac{VA\times GE}{2CW\times GQ^2}$, the ſame as in prop. v. cor. 3. & 4.

Cor. 2. If a fixed axis paſſes through the centre of gravity, and a body *A* impinges on the ſyſtem in the point *F* in a direction which is perpendicular to *GE*, *Q* being the principal centre of gyration, the reſiſtance of inertia oppoſed at *E* ‖ will be the ſame as if $\frac{W\times GQ^2}{GE^2}$ were accumulated in the point *E*, the other matter being removed: this will give the following proportion, as $\frac{A\times\S GF^2+W\times GQ^2}{GE^2} : A :: V : \frac{VA\times GE^2}{W\times GQ^2+A\times GF^2}$ the velocity of the point *E* after the impact; when *A* is very ſmall, the velocity will become $=\frac{VA\times GE^2}{W\times GQ^2}$, and the angular velocity round the centre of gravity =

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† Page 403.

* Sect. X. Prop. V.

‡ Sect. VI. Prop. VIII.

‖ Sect. VI. Prop. VIII. & Sect. X. Prop. IV. Cor. 3.

§ Page 417.

† Sect. X. Prop. V. Cor. 4.

$\frac{VA \times GE}{2CW \times GQ^2}$, † the ſame as when the ſyſtem moved in free ſpace unconfined by any axis, from which we have this general concluſion, that when a body is ſtruck in the manner deſcribed in prop. v. by a body of very ſmall inertia, the angular velocity in free ſpace will be the ſame as that which is generated round a fixed axis which paſſes through the centre of gravity; and the motion of the centre of gravity in free ſpace is the ſame as would be generated in that centre if the ſtriking body impinged directly on it, every thing elſe being the ſame.

Cor. 3. The velocity with which the point O revolves round the centre of gravity conſidered as fixed, is equal to the velocity of the centre of gravity in free ſpace: for the angular velocity of the ſyſtem revolving round the

* Cor. 2.

centre of gravity* is $\frac{VA}{2CW \times OG}$, which being multiplied into $2C \times OG$ will be $\frac{VA}{W}$ = the velocity of the point O, being the ‡ ſame as the velocity of the centre of gravity moving in free ſpace.

‡ Sect. X. Prop. V. Cor. 2.

Cor. 4. Since the centre G moves with a velocity equal to that of the point O round G conſidered as a fixed centre, it is plain that the line deſcribed by the point O, during the motion of the ſyſtem, muſt be the common cycloid.

VII.

Fig. XCVI.

Let G repreſent the centre of gravity of a body SIK moving in the direction HGI with the velocity V, each particle alſo moving with the ſame velocity V; and let a fixed obſtacle be oppoſed to any point E of the line GK, which is perpendicular to HG; it is required to aſſign the motion of the ſyſtem after the impact.

The ſyſtem SIK is conſidered to be moveable in the plane $SGIK$ only: this being premiſed, if E be made the centre of ſuſpenſion, O will be the centre of percuſſion correſponding, O will alſo be the centre of ſpontaneous* rotation: when therefore the ſyſtem impinges againſt the fixed obſtable at E, that point of the ſyſtem E will be quieſcent, and the point O not being affected by the impact, will go on with the velocity V, and conſequently the centre of gravity will proceed with the velocity $\frac{V \times EG}{OE}$; and if $C = 3.14159$, &c. the angular velocity of the ſyſtem round E, at the inſtant of the impact will be $= \frac{V}{2C \times OE}$.

* Sect. X. Prop. IV. Cor. 4.

If any other point of the line EF parallel to GI impinges againſt the immoveable obſtacle, the effect will be the ſame, provided the ſurface of the body at the point of impact be perpendicular to the direction in which it ſtrikes the obſtacle, and there be no friction or adheſion between the obſtacle and ſtriking body.

If the direction of the impact againſt the obſtacle be oblique to the ſurface of the impinging body at the point of impact, it muſt be reſolved into two, of which one is perpendicular and the other parallel to the ſurface; the latter of theſe will have no effect in altering the motion of the body ſtruck, as the ſurfaces are by the ſuppoſition perfectly ſmooth and hard; the perpendicular force will act preciſely as in the caſe of the direct impact.

Fig. XCVIII. When an irregular body LMN impinges on an irregular body SIK, ſeveral circumſtances relating to the impact are to be attended to in order to aſcertain the motion of each body after the impact: let the direction of the impact be QFS; and let the velocity of the impact be repreſented by QF; reſolve this into two, FV perpendicular to a tangent to the curve at F, and VQ parallel to that tangent. 1. At the inſtant of the impact, the body LMN may adhere firmly to the body SKI, ſo that the whole muſt move altogether as one ſubſtance: this is the caſe when a bullet of any kind penetrates the ſubſtance of an obſtacle, and after the impact becomes ſo united with it as to entirely partake of its motion. 2dly. The bodies may not be ſo united by the impact as to become one ſubſtance, but the ſurface of the ſtriking body may nevertheleſs ſo adhere to the body ſtruck, that not only the perpendicular force FV ſhall be efficient to propel it, but the lateral force alſo QV ſhall contribute to generate rotatory motion round the centre of gravity, and rectilinear motion in that centre. 3dly. When the ſurfaces are hard and ſmooth the lateral force *QV* will have no effect on the motion of the body ſtruck, which will in this caſe be impelled by the perpendicular force *VF* only.

VIII.

Fig. XCVIII.

Let a body *LMN* impinge on the body *MIH* at reſt, the point of impact being at *F*, and the direction of the impact *QS*: it is required from the neceſſary conditions to aſcertain the motion of the two bodies after the impact.

Let *G* be the maſs of the ſtriking body, *I* the maſs of the body ſtruck: let *QF* repreſent the velocity of the impact in quantity and direction: this being oblique to the ſurfaces at *F* muſt be reſolved into two *FV*, *QV*, the former perpendicular and the latter parallel to the curves at *F*: the ſurfaces of the bodies being perfectly hard and ſmooth, the force *VF* only will be effectual in impelling the body ſtruck. Produce *VF* indefinitely: and through the centres of gravity *G* and *I* draw *OGE*, *KIH* perpendicular to *VFH*: and ſuppoſe the body *G* to vibrate round an axis which paſſes through *E*, let *O* be the centre of percuſſion; and when the body *I* revolves round *H*, let *K* be the centre of percuſſion. It is here to be obſerved, that becauſe the ſurfaces at *F* are not ſuppoſed to adhere, the motion of the body *I* will not be impeded by the inertia of the ſtriking body: the force *VF* impelling the body *I* in the direction *FH*, will cauſe it to begin its motion round the centre of ſpontaneous rotation *K*, while *F* deſcribes an evaneſcent arc of which the radius is *KF*; but the ſtriking body by its inertia will endeavour to proceed in the direction *FH* until the bodies have a common velocity in that direction; this will manifeſtly cauſe the ſurfaces to ſlide over each other near *F* through a ſmall ſpace, but the time in which the impact is performed being in a phyſical ſenſe an inſtant, the force will during the whole of the time be applied to move the point *H*: this being premiſed, let the perpendicular velocity of impact *VF* be repreſented by *V*: through *G* and *I* draw *CGL*, *MIX* parallel to *VFH*; then will the point *I* deſcribe the line

IX[‖] after the impact, and G will deſcribe the line GL, (the lateral motion QV being conſidered afterwards:) the reſiſtance of inertia oppoſed by the body I, is the ſame as if the whole maſs were removed, and the equivalent maſs $\frac{I\times KI}{KH}$ were [*]concentrated in H; and the ſyſtem G which impinges on I will ſtrike that body with the ſame force as if the whole maſs in G were removed and the equivalent[‡] maſs $\frac{G\times OG}{OE}$ were concentrated in E: wherefore the laws of colliſion will give this proportion, as $\frac{I\times KI}{HK} + \frac{G\times OG}{EO} : \frac{G\times GO}{OE} :: V :$ to the velocity communicated to H, which being diminiſhed in the proportion of HK to KI will be the velocity of the centre of gravity in the direction IX: moreover, the velocity of the centre of gravity and its diſtance from the centre of ſpontaneous rotation will give the angular velocity of the ſyſtem about K or I. And becauſe O is the centre of ſpontaneous rotation of the body G, that centre [†]will not be at all affected by the impact, and the velocity of the point F in the ſtriking body eſtimated in the direction of the perpendicular FV will be the ſame with that which is communicated to H: let this latter velocity be expounded by the line EU when OT repreſents the original perpendicular velocity of impact, and join TUW, this will give the perpendicular velocity of the centre of gravity $= Gg$, and the angle deſcribed by the body G in its rotation during the ſame time $= TWO$ being the meaſure of the angular velocity of the ſtriking body after the impact. The velocity of the centre of gravity Gg in the direction GL, was by conſtruction to the entire perpendicular velocity of impact as Gg to TO; we have therefore the perpendicular velocity of G after the impact in the direction $GL = V \times \frac{Gg}{TO}$, which being compounded with the lateral motion of G denoted by QV, will give the true velocity and direction of the centre G after the impact.

[‖] Sect. X. Prop. VI.

[*] Sect. VI. Prop. VIII. & Sect. X. Prop. IV. Cor. 6.

[‡] Sect. X. Prop. IV. Cor. 2.

[†] Sect. X. Prop. IV.

IX.

Fig. XCVI. In any ſyſtem *SIK* to which when quieſcent, motion has been communicated by the impulſe of a force without inertia; that is, rectilinear motion to the centre of gravity *G* meaſured by the ſpace *D*, which the point *G* would deſcribe uniformly in any given time, and rotatory or angular motion meaſured by the number *M* revolutions, which it would deſcribe uniformly round *G* in the ſame given time: it is required to aſcertain at what perpendicular diſtance *GE* from the centre of gravity, the direction of the impelling force *LE* muſt have paſſed, ſo as to have generated the rectilinear and rotatory motions deſcribed in the propoſition.

Let *LE* be the direction of the impulſe, and through *G* the centre of gravity of the ſyſtem, draw *OGE* perpendicular to *LE*, and let *O* be the centre of ſpontaneous rotation of the ſyſtem when impelled at *E* or *F*. Let *Q* be the principal centre of gyration of the ſyſtem when it revolves round the centre of gravity. Moreover, let *A* be a body which impinges on *F* with a velocity *V*, *W* being the weight of the ſyſtem: then the velocity communicated to the centre of ‡gravity $= \frac{VA}{W}$, and the angular velocity generated in the ſyſtem round *G* is $= \frac{VA}{2CW \times OG}$, or $\frac{VA \times GE}{2CW \times GQ^2}$†, *C* being = 3.14159, &c. but by the problem, the velocity communicated to the centre of gravity is *D*, and the angular motion is *M*; that is, the num-

‡ Sect. X. Prop. V. Cor. 2.

† Sect. X. Prop. III. Cor. 3.

number of revolutions or parts of a revolution deſcribed while the centre of gravity deſcribes the ſpace $D = M$: we have therefore from the conditions, this equation: $\frac{D \times GE}{2C \times GQ^2} = M$: and the diſtance ſought or $GE = \frac{2CM \times GQ^2}{D}$.

The earth revolves about an axis paſſing through its centre of gravity, while that centre is carried on in free ſpace deſcribing an orbit nearly circular. This double motion has been diſcovered in ſeveral of the planets, and by analogy may be ſuppoſed to exiſt in the others, although their diſtances from us, unvaried ſurfaces, or different cauſes have prevented any ſatisfactory obſervation of their revolutions round their axes.

Moſt of the other phænomena in the planets' motions are acknowledged to be intimately related, and dependant on each other, and to be regulated by certain definite laws, from which they deviate not in the ſmalleſt degree; yet the rotation of the planets round their axes, as far as they have been obſerved ‡, ſeem to follow no rule or order whatever, either in reſpect of their diſtances from the ſun, their periodic times, quantities of matter, or any other circumſtance of appearance or motion. This is no more, however, than ſaying that no law has been hitherto diſcovered to which theſe phænomena can be referred; but to argue from hence, that no law exiſts in nature by which the rotations of the planets round their axes were primarily adjuſted, would be to acknowledge, that this part of the ſolar ſyſtem is exempt from that harmony and order, which are ſo conſpicuous in the other appearances relating to the planets' motions. The propoſition above demonſtrated immediately applies to the ſubject in queſtion, at firſt ſight ſuggeſting the probability of diſcovering ſome proportion or relation between the rotation of the planets and their other motions, their diſtances from the ſun, &c. and although little at preſent is collected from the examination, yet when the revolutions of the planets round their axes are more generally and more exactly known, and other phænomena combined with them, future experience may perceive, that the times of rotation could have been no other than what they are, according to the known laws of motion.

‡ J. Bernoulli. Vol. IV. p. 280.

To apply the propoſition above demonſtrated to the motion of the earth. Let $SIKG$ repreſent a ſection of the earth paſſing through its centre of gravity G, which

Fig. XCIX.

conſidering the earth as ſpherical and homogeneal will be the centre of the ſphere. Suppoſe the earth to have been originally quieſcent, and that an impulſe was impreſſed on it in the direction *CGA*; if the direction of this impulſe paſſed through the centre of gravity *G*, no rotation round an axis would have been the conſequence; if the impulſe paſſed in the ſame direction, but not through the centre of gravity, a motion of rotation ‡ round the centre *G* would be generated while the centre of gravity proceeded in the right line *GA* parallel to the direction of the impelling force. It is manifeſt, that the attraction of the ſun which cauſes the centre *G* to deviate from the right line *GA*, and to deſcribe an arc *GH* (that may be here aſſumed as circular without error) will alter neither the velocity of the centre *G* nor the motion of rotation. Through *G* draw *OGEB* perpendicular to *LE*. Having given the magnitude of an impulſe relatively to the earth's maſs, impreſſed in the direction *LE*, and the diſtance *GE* on the earth's radius from the centre of gravity at which it was applied, the abſolute velocity of the centre of gravity *G*, and the angular motion round *G* would be inferred*; and converſely according to the conditions of the propoſition above demonſtrated, if the angular motion and the abſolute velocity of the centre of gravity be given, the diſtance *GE* at which the impulſe was applied from the earth's centre will be known: to apply this, let *GQ* be the diſtance of the principal centre of gyration from the earth's centre: let the earth's radius be $= 1$, being aſſumed as a ſtandard of ſpace; and let the earth's periodic time round its axis, or $23^h.\ 56'.\ 4''$. be $=1$, a ſtandard † of time, being a ſidereal day; in order to expreſs the velocity of the earth's centre of gravity, we muſt determine the ſpace deſcribed by it in one ſidereal day; and ſince the ſun's parallax is about $8\frac{1}{2}''$, the circumference of the earth's orbit will contain 152470 of the earth's radii, and becauſe this ſpace is deſcribed by the earth's centre in 365.24 mean days or 366.24 ſidereal days, the ſpace deſcribed by the earth's centre in one ſidereal day will be $=\frac{152470}{366.24}=416.31$ of the earth's radii $= D$ by the propoſition, which will be the earth's velocity in its orbit referred to the time of one ſidereal day. We have therefore in the equation $GE=\frac{2CM\times GQ^2}{D}$, $C=3.14159$, $M=1$ being the angular velocity of the earth round its

‡ Sect. X. Prop. VI.

* Sect. X. Prop. V. Cor. 2 & 3.

† Sect. II. Prop. VII.

axis, or the number of revolutions deſcribed while the centre G paſſes through the ſpace D: moreover GQ^2 being the ſquare of the diſtance of the principal centre of gyration from the centre of gravity $= \frac{2}{5} \times$ ſquare*of the radius, or ſince the radius $= 1$, $GQ^2 = \frac{2}{5}$, and $D = 416.31$, which being ſubſtituted, we ſhall have $GE = \frac{2 \times 3.14159 \times \frac{2}{5}}{416.31} = \frac{1}{165.6}$. From hence it appears, that if an impulſe be impreſſed on a quieſcent ſphere, and the direction of the force ſhould be at a perpendicular diſtance from the centre of gravity of $\frac{1}{165.6}$ part of the radius: the angular motion of the ſphere, and the abſolute motion of the centre, will be proportional to thoſe of the earth. From this determination, the earth's centre of ſpontaneous rotation is obtained: for making O that centre, we have † $GO = \frac{GQ^2}{GE} = \frac{2}{5} \times 165.6 = 66.25$, which is ſomewhat more than the moon's greateſt diſtance from the earth's centre.

Bernoulli makes ‡ the diſtance $GE = \frac{1}{150}$ and $GO = 60$, from aſſuming the ſun's parallax $9''.375$, which is now known not to exceed $8\frac{1}{2}''$.

In the ſame manner the diſtances GE and GO may be aſcertained for the other planets, their times of revolution round their axes and the abſolute velocities of their centres of gravity being known.

* Sect. VI. Prop. XII. p. 225.

† Sect. VI. Prop. VIII.

‡ J. Bernoulli, Vol. IV. p. 283.

X.

GIF repreſents a ſyſtem exiſting in free ſpace: ſuppoſe that a body A impinges on the point I in the direction HI with the velocity V, and that at the ſame inſtant another body B impinges on the point F in any other direction LF with the

Fig. C.

the velocity U; let it be required to determine the rectilinear motion of the centre of gravity G after the impact, and the angular velocity generated in the ſyſtem, ſuppoſing that the inertia of the ſtriking bodies is inconſiderable.

Through the centre of gravity G draw GC parallel to HI, and GD parallel to LF; make GC to GD in the proportion compounded of the ratio of $A : B$, and of $V : U$; compleat the parallelogram, and draw the diagonal GK: then will GK be the direction in which the centre of gravity G moves after the impact, its velocity being ſuch as will carry it uniformly over the ſpace GK in the ſame time that GC would be deſcribed by it, if the body A only impinged on the point I in the direction HI with the velocity V. Produce HI and LF indefinitely, and through G draw GP perpendicular to HIP, and GE perpendicular to LFE. Let Q be the principal centre of gyration, W the maſs of the body: then referring the velocities to any ſtandard of time 1, the angular velocity generated by the impact of the body A, will be that which would cauſe the ſyſtem to deſcribe* $\frac{VA \times GP}{2CW \times GQ^2}$ revolutions in the time 1, C being the number 3.14159, &c. And the impact of the body B if applied alone would generate an angular velocity of $\frac{UB \times GE}{2CW \times GQ^2}$ revolutions in the ſame ſtandard time 1; and if the two impacts at F and I act to turn the ſyſtem round G contrary ways, the angular velocity after the impact will on the whole be $\frac{VA \times GP - UB \times GE}{2CW \times GQ^2}$ or $\frac{UB \times GE - VA \times GP}{2CW \times GQ^2}$ according as $VA \times GP$ is greater or leſs than $UB \times GE$.

* Sect. X. Prop. V. Cor. 4.

XI.

Fig. CI. G repreſents the centre of gravity of a body moving in free ſpace with the velocity V; and at the ſame time revolving round G with an angular velocity of M revolutions in the ſtandard time to which V is referred; the maſs of the ſyſtem is W, and Q is the principal centre of gyration: it is required to aſſign the velocity with which a body A muſt impinge on any given point K, ſo that the point K may be quieſcent the inſtant after the impact, it being ſuppoſed that the direction of the impact NK is perpendicular to KG paſſing through the centre of gravity, and that the inertia of A is inconſiderable in compariſon of W.

It is manifeſt from the conſtruction in prop. ii. that if a line KG be drawn through the centre of gravity perpendicular to the direction of impact, or which is the ſame thing, perpendicular to the direction in which the centre of gravity moves, the centre of ſpontaneous rotation will be ſomewhere in this perpendicular line KG produced.

Let O be the centre of ſpontaneous rotation, then becauſe V is the velocity of the centre of gravity, and M is angular velocity[*], $GO = \frac{V}{2C \times M}$, C being $= 3.14159$, and $GE = \frac{GQ^2 \times 2CM}{V}$, Moreover ſince V is the velocity of the point G, $\frac{V \times OK}{QG}$ will be the velocity of the point K.

Let T be the centre of ſpontaneous rotation correſponding to the point K: the force of the ſtroke is the ſame as if an equivalent maſs, $=$ the whole maſs diminiſhed in the proportion of $KT : GT$[†], being concentrated into a point, impinged directly on it with the ſame velocity.

To ſolve the problem therefore it will be ſufficient to aſſign the velocity of the body A, when $A \times$ into that velocity ſhall be equal to the equivalent maſs into the velocity of its motion. Let z be the velocity of A required:

[*] Sect. X. Prop. V. Cor. 2 & 3.

[†] Sect. X. Prop. IV. Cor. 2.

quired: the velocity of K is $\frac{V\times KO}{GO}$, and becauſe T is the centre of ſpontaneous rotation correſponding to K, the equivalent maſs to be concentrated into K, ſo as to cauſe a reſiſtance of inertia equal to that of the whole ſyſtem is $\frac{W\times GT}{KT}$, from which the following equation is ‖ Sect. VI. Prop. VIII. derived: $Az = \frac{V\times KO\times W\cdot GT}{GO\times KT}$, but ‖ $GT = \frac{GQ^2}{GK}$, $GO = \frac{GQ^2}{GE}, = \dagger \frac{V}{2CM}$ and $KO = GK + \frac{GQ^2}{GE} = GK$ † Sect. X. Prop. V. Cor. 2 & 3. $+ \frac{V}{2CM}$, and $KT = GK + \frac{GQ^2}{GK}$, wherefore $Az = \frac{V\times W\times\overline{2CM\times GK+V}\times GT\times GK}{2CM\times GO\times\overline{GK^2+GQ^2}}$; but $2CM\times GO = V$, and $GT\times GK = GQ^2$, we have therefore $Az = \frac{W\times GQ^2\times\overline{V+2CM\times GK}}{GK^2+GQ^2}$, and the velocity acquired, or

$$z = \frac{W\times GQ^2\times\overline{V+2CM\times GK}}{A\times\overline{GK^2+GQ^2}}.$$

Cor. 1. If the centre of gravity moves in the direction GI with the velocity V, the angular velocity being M, and any point K impinges on an oppoſed obſtacle when the line GK is perpendicular to GI, the force of the ſtroke will be

$$Az = \frac{W\times GQ^2\times\overline{V+2CM\times GK}}{\overline{GK^2+GQ^2}}.$$

Cor. 2. While the ſyſtem is moving in free ſpace in the manner deſcribed in the propoſition, if a body impinges on any point F, not in the line GK, which is perpendicular to the impact, it is manifeſt, that the velocity cannot be ſo adjuſted ſo that this point ſhall be motionleſs immediately after the impact, for if the point F were quieſcent, H the centre of ſpontaneous rotation correſponding to it, would move in a direction perpendicular to HF; but from what has been demonſtrated, it muſt proceed in the direction perpendicular to OK, in which it was moving at the time of the impact.

XII.

XII.

Let *WTEF*, *QBGF* be two ſyſtems revolving in free ſpace about their centres of gravity *G* and *W*; let the angular velocity of the ſyſtem *QBGF* be ſuch as cauſes it to make *M* revolutions or parts of a revolution in any ſtandard time 1, and in the ſame time let the ſyſtem *WTEF* perform *N* revolutions or parts of a revolution in the ſame plane, but in a direction contrary to that in which the former ſyſtem revolves: moreover, let the centre of gravity *G* move in the right line *XB* with the velocity *V*, and let the centre *W* move in the oppoſite direction *WD* parallel to *BX*: ſuppoſe the ſyſtem *QBGF* to impinge on the body *WTEF* in the point *F* and in the direction *LF* parallel to *HD* or *BX* and perpendicular to the ſurface which it ſtrikes; it is required to aſſign the motion of each body after the impact. Fig. CII.

Through *G* and *W* draw *YL* and *OE* perpendicular to the direction in which the centres of gravity move, and let the principal centre of gyration of the ſyſtem *QBGF* be *Q*. and that of the ſyſtem *WTEF* be *T*. Moreover, let the maſs of the ſyſtem *QBGF* be *G*, and that of the other ſyſtem *W*: the point *F* in the ſyſtem *WTEF* will be impelled by the impact with the ſame force as if the maſs $\frac{G \times GQ^2 \times \overline{V + 2CM \times GL}}{\text{velocity of } L \times \overline{GQ^2 + GL^2}}$ [*] impinged againſt *F* being concentrated into a point with the ſame velocity, on a ſuppoſition that the inertia of the ſtriking body at

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[*] Prop. XI. Cor. 1.

F is evaneſcent or inconſiderable. But becauſe the angular velocity of the ſtriking body is M, and the velocity of its centre of gravity $= V$, the velocity of L is $= V + 2MC \times GL$: the equivalent maſs will therefore be $\frac{G \times GQ^2}{GQ^2 + GL^2}$ impinging with the velocity $V + 2MC \times GL$. This impact will generate in the centre of gravity W a * velocity $= \frac{G \times GQ^2 \times \overline{V + 2MC \times GL}}{W \times \overline{GQ^2 + GL^2}}$ in the direction WH: wherefore after the impact the velocity of the centre of gravity W in the ſame direction will be $V - \frac{G \times GQ^2 \times \overline{V + 2MC \times GL}}{W \times \overline{GQ^2 + GL^2}}$: moreover, the ſame force of impact will generate an angular velocity in the ſyſtem *$WTEF = \frac{G \times WE \times GQ^2 \times \overline{V + 2MC \times GL}}{2C \times W \times WT^2 \times \overline{GQ^2 + GL^2}}$, which being added to the angular velocity before the impact N or ſubtracted from it, according as the impact tends to augment or counteract the rotation of the body ſtruck, the angular velocity of the ſyſtem $WTEF$ will be obtained.

* Sect. X. Prop. V. Cor. 2.

* Cor. 3.

Let A repreſent the angular velocity of the ſyſtem $WTEF$ immediately after the impact which has been juſt determined, then if C be $= 3.14152$, &c. $2CA \times WE$ will be the velocity of the point E, or of the point F in the ſtriking body: for according to the reaſoning contained in page 422*, when the ſurfaces at F are perfectly ſmooth the point F of the ſtriking body will proceed at the inſtant after the impact in the direction FE, until the two bodies have a common velocity in the direction FE: in the line LE take LI equal to the ſpace which would be deſcribed by the point E with the velocity $2CA \times WE$ in any ſtandard particle of time.

* Ad finem.

Let Y be the centre of ſpontaneous rotation in the ſyſtem $QBGF$: the motion of this point will not be affected by the impact; its velocity and direction therefore will be the ſame after the impact as it was before the ſyſtem $QDBGF$ impinged on the other body: becauſe the angular velocity of the ſtriking body is M, in the direction from L to F, and the velocity of the centre of gravity V in the direction GB, the velocity of the point Y before the impact will be $V - 2CM \times GY$ or $V - \frac{2CM \times GQ^2}{GL}$: make YZ

perpendicular to LY and equal to the ſpace through which point Y is carried uniformly by the velocity $V - \frac{2CM \times GQ^2}{GL}$ in the ſtandard particle of time; through I draw IZP interſecting GB in g and LG produced in P; then Gg will be the ſpace which the centre of gravity G will deſcribe uniformly after the impact in the particle of time 1, and will therefore repreſent the velocity of the centre of gravity G after the impact: and IPL will be the angle deſcribed in the ſame time, and will therefore be a meaſure of the angular velocity of the ſtriking body after the impact.

Cor. 1. If $\frac{G \times WE \times GQ^2 \times \overline{V + 2MC \times GL}}{2C \times W \times WT^2 \times \overline{GQ^2 + GL^2}}$ be leſs than N, the rotation of the ſyſtem $WTEF$ will be in the ſame direction after as before the ſtroke. If the two quantities juſt mentioned are eqnal, the body ſtruck will have no angular velocity after the ſtroke, and if the former ſhould exceed the latter, the direction in which the ſyſtem revolved will be changed by the impact.

Cor. 2. If $LI = zY$, the ſtriking body will not revolve after the impact, its angular velocity being deſtroyed. If LI exceeds zY, the rotation will be continued in the ſame direction after the impact as before it: if zy exceeds LI the direction of rotation will be changed by the impact.

APPENDIX.

CONTAINING A DESCRIPTION OF THE ADJUSTMENTS AND PRACTICAL USE OF THE MONOCHORD REPRESENTED IN FIG. XIX. Vid. p. 99.

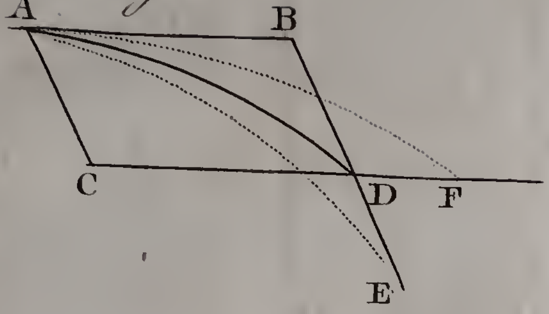
IN this conſtruction the ſtring is not terminated by a bridge, but between metallic edges, which not only define the ſtring's length with great preciſion, but prevent any irregularity in the tending force, which is the unavoidable conſequence of bridges: the two ſteel edges at *C* are ſeparated from each other by the ſcrew *F*, in order to give room for the ſtring, while the frame *TQRS* is moved along the ſcale. When the ſcrew *N* is looſened, the frame *TQRS* moves freely along the ſcale, carrying with it the ſteel edges, &c. By turning the ſcrew *N*, the frame *TQRS* is faſtened to the ſcale, while the other parts *OGDC* are moveable only by means of the ſcrew *EWP*, which ſerves the purpoſe of adjuſtment as well as for other uſes hereafter mentioned. *KL* is a part of the braſs ſcale *AB* repreſented in the left hand figure, the whole length is about 40 inches, and the breadth about $\frac{1}{2}$ an inch. In this ſcale 100 equal parts are ſet off, and each part is biſected by a ſhorter diviſion. In the ſquare aperture *M*, an index is engraved the diſtance of it from the ſteel edges, eſtimated in the direction of the ſcale, is ſuch, that when this index is ſet to any diviſion, the length of the ſtring intercepted between the ſteel edges *C*, and the upper termination of the ſtring, may be exactly equal to the number of parts expreſſed by the diviſion. Suppoſe the frame *TQRS* to be faſtened, the ſcrew *EWP* being turned twenty revolutions, carries the ſteel edges, or the index at *M*, through a ſpace equal to one whole diviſion, and conſequently one revolution of the ſcrew carries the ſteel edges through one twentieth part of a diviſion.

The circumference of the ſcrew is divided into 50 equal parts, each of which muſt therefore anſwer to one thouſandth part of a diviſion, that is, if the whole length of a ſtring be made = 100 parts, one of the diviſions on the ſcrew's circumference will anſwer to the hundred thouſandth part of the whole length. By this means, it will be eaſy to adjuſt the length of ſtring intercepted between the ſteel edges *C* and the upper extremity, of any magnitude leſs than 100000: but becauſe any muſical interval implies the vibrations of two ſtrings, of which the lengths are in the direct proportion of the times of vibration, a baſe note is always aſſumed as a ſtandard to which the others are referred. In order therefore to exemplify the uſe of this inſtrument, the length of the baſe note muſt be firſt adjuſted. The length of the ſtring which conſtitutes this baſe note, is moſt conveniently aſſumed = 10, 100, 1000, or ſome number in that progreſſion. In this inſtrument, the number correſponding to the baſe note is 100000: after having ſeparated the ſteel edges at *D* by the ſcrew *F*, bring the index in the ſquare aperture to the diviſion on the ſcale marked 100, as near as can be judged by the naked eye, and fix the frame *TQRS* by turning the ſcrew marked *N*. Then to adjuſt the coincidence of index preciſe, apply the microſcope *G* immediately over the index, and moue the ſcrew *EWP* until the index coincides with the diviſion 100: the two edges being now ſuffered to compreſs the ſtring, the note ſounded by it will be the baſe or fundamental tone. Now, let it be required to adjuſt the length of the ſtring = 66667 parts, the whole length being 100000. The operation will be as follows; looſen the ſcrew *N* that the frame *TQRS* may move freely: bring the index on the ſquare aperture to 66 on the ſcale, and adjuſt by the glaſs as before; bring the moveable index *x* to the point 0 on the circumference *WEP*: then, ſince the third figure in the number propoſed is 6, anſwering to 6 thouſandths of the whole length, this will be ſet off by turning the ſcrew *EPW* twelve revolutions: by which the three firſt figures 666 will be expreſſed in the length of the ſtring. Moreover, as there are 67 hundred thouſandth parts remaining to be added to the length, and one of the diviſions on the ſcrew's circumference anſwers to one hundred thouſandth part, it follows, that the ſcrew muſt be turned through 67 of theſe diviſions; that is, becauſe the circumference of the ſcrew is divided into 50 equal parts, the ſcrew muſt be turned one revolution and 17 diviſions. Very little time is taken in making theſe adjuſtments, the uſe of which is to define the ſtring to any given length and true to the hundred thouſandth part of the whole length; by which the different ſyſtems of muſical ſounds, which have been before invented, ſuch as Huygens's, Ptolemy's, Smith's, &c. may be brought into compariſon, as well as others which may be ſuggeſted either from theory or trials. The monochords of the old conſtruction, were divided according to one or two given ſcales only, which rendered them wholly uſeleſs in making experiments on harmonic temperaments in general, for which purpoſe the inſtrument above deſcribed was conſtructed.

The method of terminating the ſtring between metallic edges, in preference to that of ſupporting it by bridges, was copied from the monochord of a very ingenious and worthy friend the Rev. Mr. L. Huddleſtone, of the Univerſity of Oxford.

ERRATA.

PAGE 3. line 14. *for* its diſtances, *read* the diſtances; p. 7. l. 19. *after* deſcend, *add* from reſt; p. 8. 3d l. from the bottom, *f.* in a direction, *read* in the direction; p. 14. l. 14. *f.* $\frac{B}{b} = 1$, *r.* $\frac{B^m}{b^m}$; ibid. laſt l. *f.* and if W, *r.* if W; p. 18. l. 2. *after* impelled, *inſert* or reſiſted; p. 29. l. 3. *f.* a thouſandth part, *r.* in thouſandth parts; p. 50. l. 19. *f.* O, *r.* B; p. 52. in the references, *f.* prop. XI. *r.* prop. XII; p. 62. l. 26. *f.* and the force, *r.* the force; p. 68. l. 11. *f.* F, *r.* T; ibid. l. 14. *f.* $a^{\frac{3}{2}}$, *r.* $p\,a^{\frac{3}{2}}$; p. 73. l. 4. *f.* the velocity, *r.* the finite velocity; ibid. l. 13. *dele* of it; ibid. l. 14. *f.* $\text{Log}\,\frac{a+x}{a}$, *r.* $a \times \text{Log}\,\frac{a+x}{a}$; p. 74. l. 16. *f.* $\dot{v} = x\,\dot{w}$, *r.* $\dot{x} = x\,\dot{v}$; p. 79. l. 25. *f.* in the length, *r.* on the length; p. 83. l. 15. *f.* $-Fa^{r+1}$, *r.* $-Fa^{n+1}$; p. 86. l. 20. *f.* $\frac{1}{1-n}$, *r.* $\frac{1}{1+n}$; p. 87. l. 4. *f.* a logarithm, *r.* a logarithmitic expreſſion; p. 92. l. 20. *f.* PO, *r.* RO; p. 95. l. 5. *f.* or one ſide, *r.* on one ſide; p. 96. l. 13. *after* ſtring vibrates, *inſert* ſhould vary; p. 97. l. 14. *f.* the diſtance, *r.* the greateſt diſtance; p. 102. l. 40. *f.* Holder, *r.* Harrington; p. 109. l. 18. *f.* HH, *r.* HO; ibid. *f.* Fig. XXII, *r.* Fig. XXIII; p. 118. l. 8. *f.* oppoſed, *r.* oppoſe; p. 126. l. 3. *f.* equal, *r.* equals; p. 135. l. 9. *f.* how nearly, *r.* how near; p. 141. l. 18. *f.* 13780, *r.* 13760; p. 143. l. 6. *f.* body's weight, *r.* body's weight in the fluid; p. 148. l. 7. *f.* on the contrary arguments, *r.* on the contrary, arguments; p. 152. l. 4. *f.* 2301585, *r.* 2302585; p. 155. l. 32. *dele* is; p. 156. l. 6. *f.* from, *r.* after; p. 160. *f.* XXX, *r.* XXXI; ibid. l. 4. *f.* 34, *r.* 33; p. 173. l. 5. *f.* Clarke, *r.* Clare; ibid. l. 33. *f.* rarer, *r.* denſer; p. 182. l. 7. *f.* degree, *r.* degrees; p. 193. ult. *f.* weight, *r.* maſs; p. 199. l. 22. *f.* Diminution, *r.* The diminution; p. 214. l. 22. *f.* by Gallileo, *r.* from Gallileo's principles; p. 268. l. 11. *f.* SD^2, *r.* SH^2; p. 269. l. 11. *f.* 10 : 1, *r.* 1 : 10; p. 281. l. 1. *f.* $\frac{Wr^2}{a^2}$, *r.* $\frac{Gr^2}{a^2}$; p. 283. l. 4. *f.* x, *r.* z; ibid. l. 6. *f.* $V^2\dot{y}$, *r.* $V^2y\dot{y}$; p. 286. l. 11. *f.* ſame, *r.* contrary; p. 290. l. 15. *after* ſtring's length, *inſert* eſtimated from C to D; p. 301. l. 5. *dele* which; ibid. *dele* of the; p. 331. l. 9. *after* will, *inſert* be; p. 335. l. 13. *f.* inſtruments, *r.* inſtrument; p. 337. l. 5. *f.* $3\frac{3}{4}$ oz. *r.* $2\frac{3}{4}$ oz. p. 345. l. 2. *f.* 239, *r.* 339; p. 355. l. 5. *f.* centre, *r.* principal centre; p. 356. l. 5. *f.* meaſures, *r.* a meaſure; p. 363. l. 2. *dele* a; p. 362. l. 25. *f.* $S : s$, *r.* $s : S$; p. 366. l. 19. *dele* is; p. 367. l. 2. *f.* accelerated, *r.* uniformly accelerated; p. 384. l. 1 *f.* 25.25, *r.* 2.525; p. 404. l. 12. *f.* weight, *r.* inertia; p. 1415. l. 14. *dele* immoveable.



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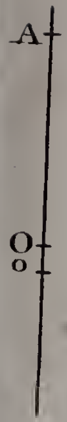

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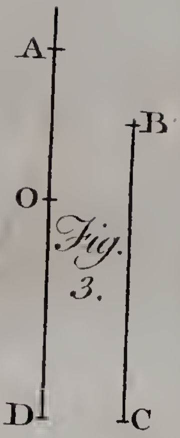

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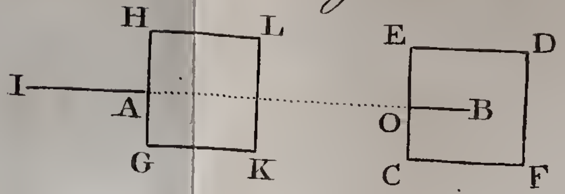

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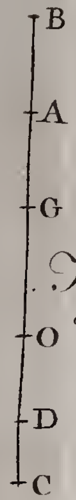

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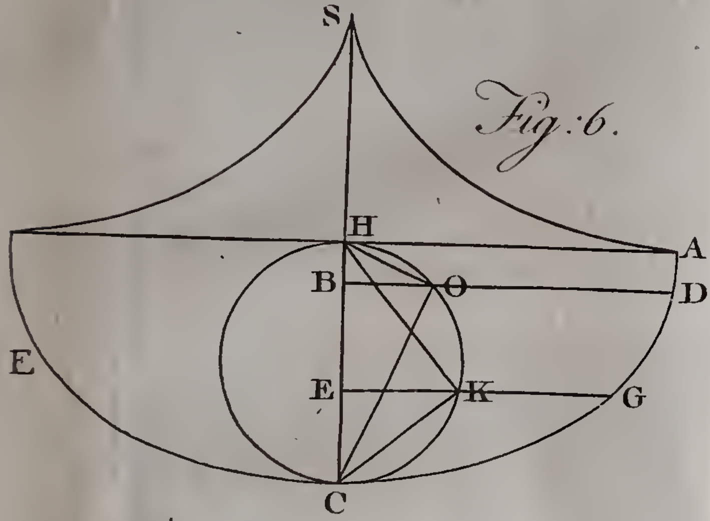

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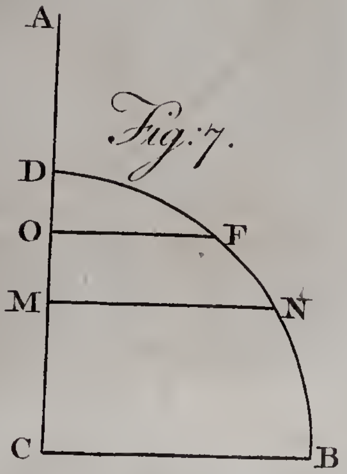

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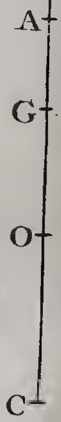

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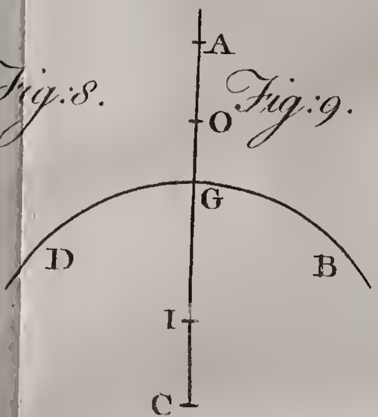

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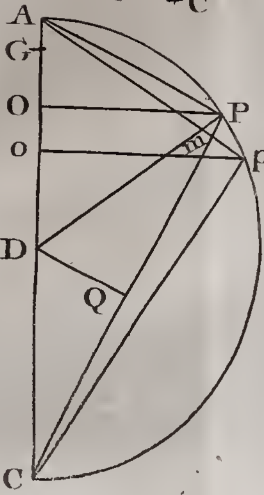

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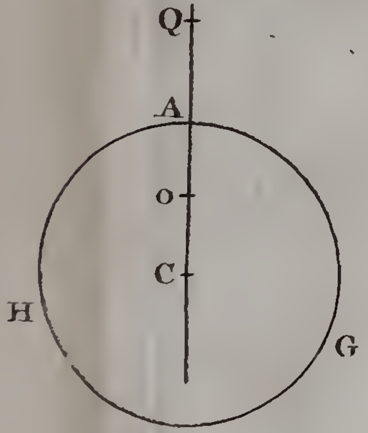

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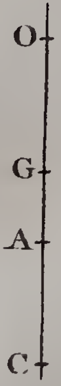

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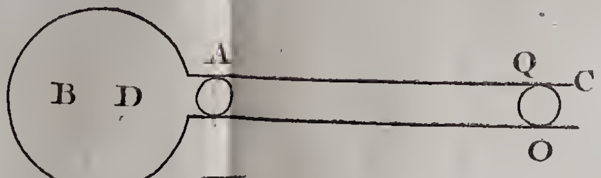

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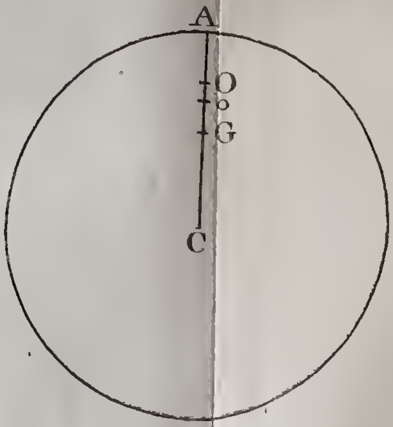

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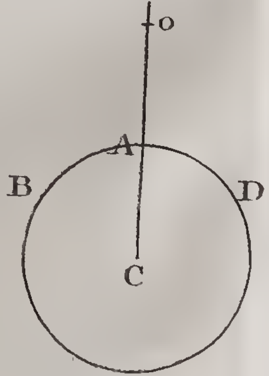

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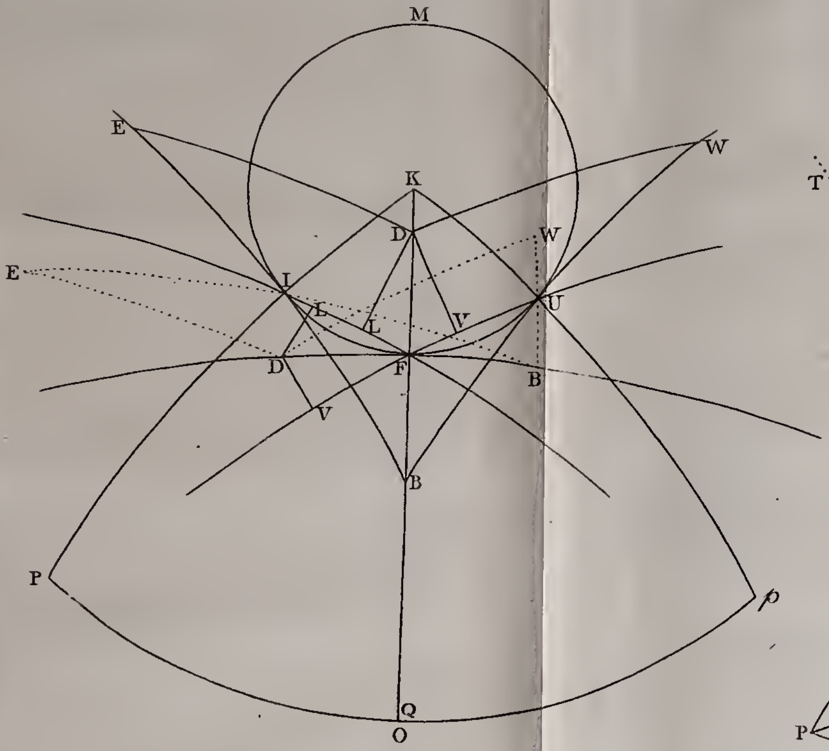
Fig. IV.



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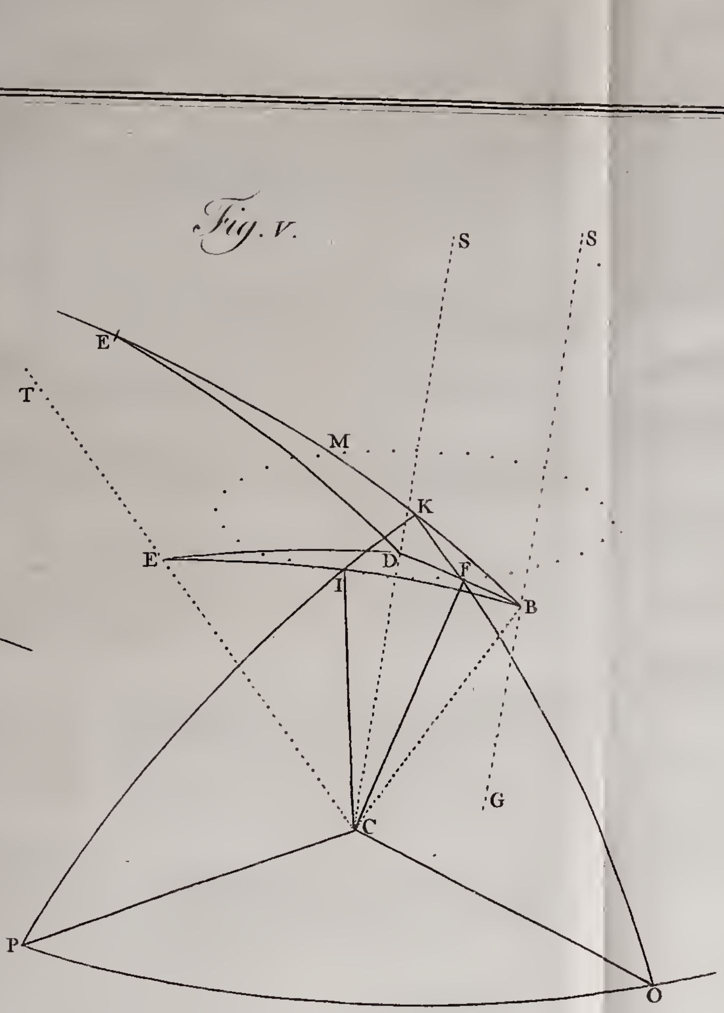

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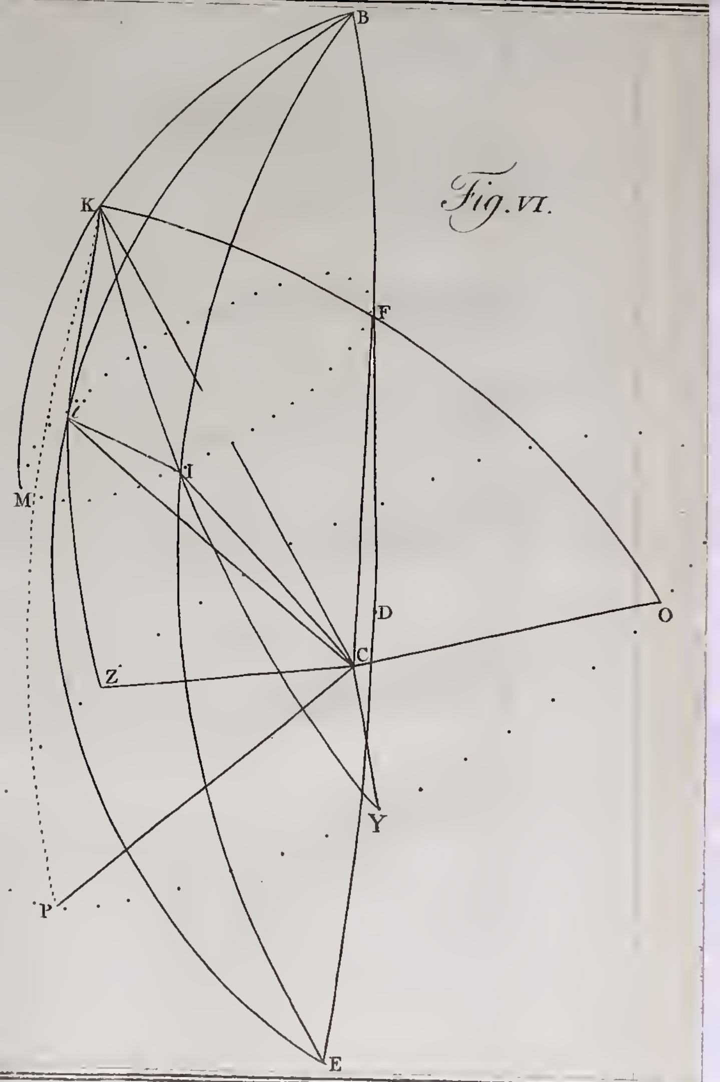

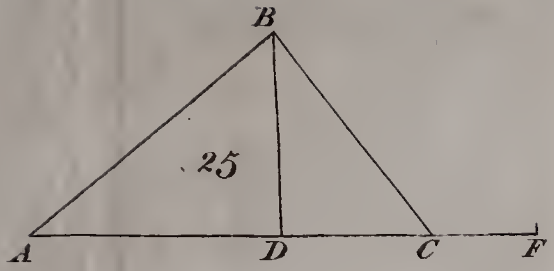
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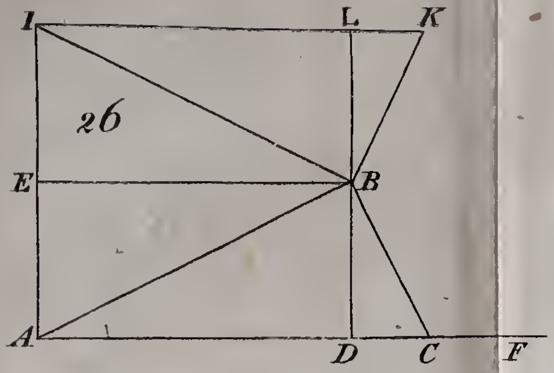
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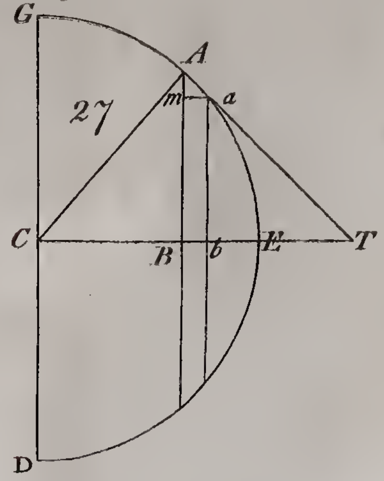
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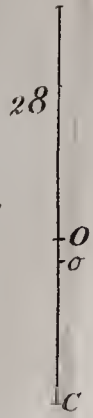
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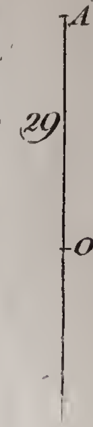
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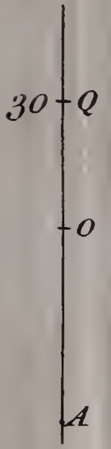
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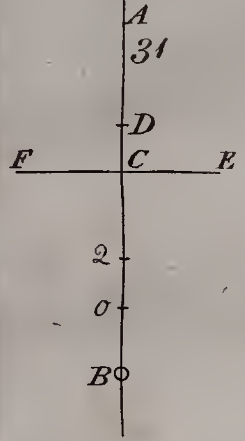
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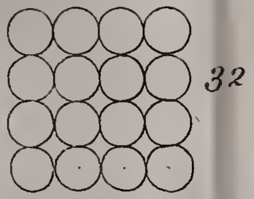
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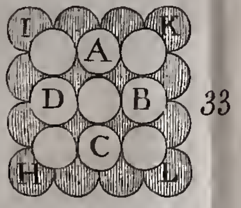
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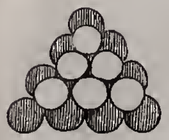

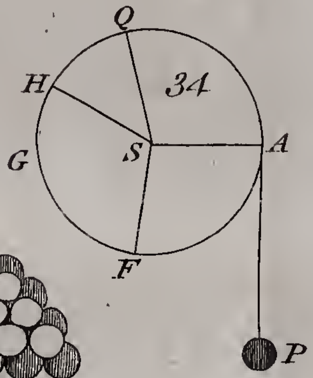
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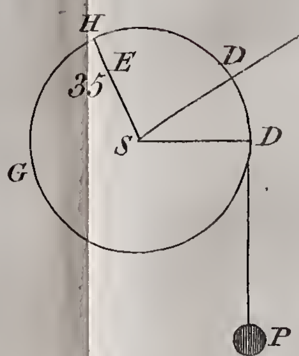
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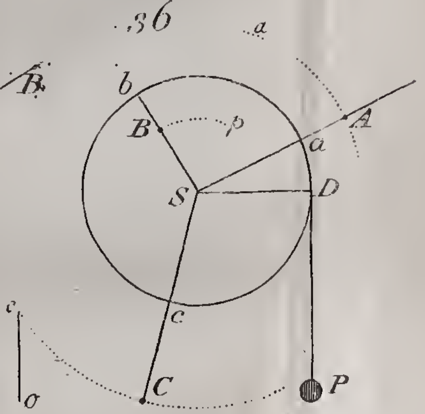
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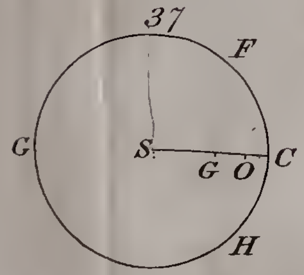
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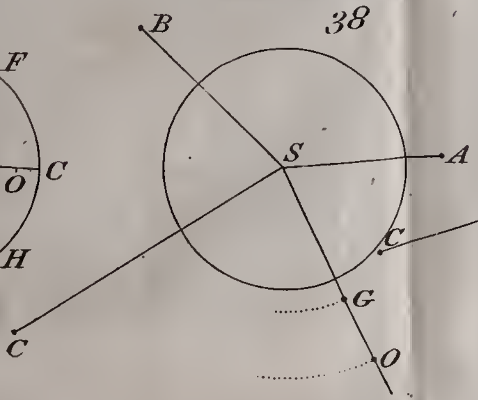
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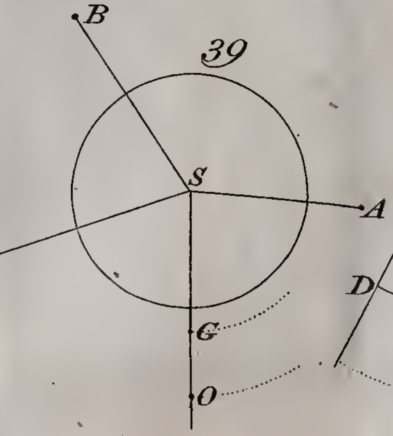
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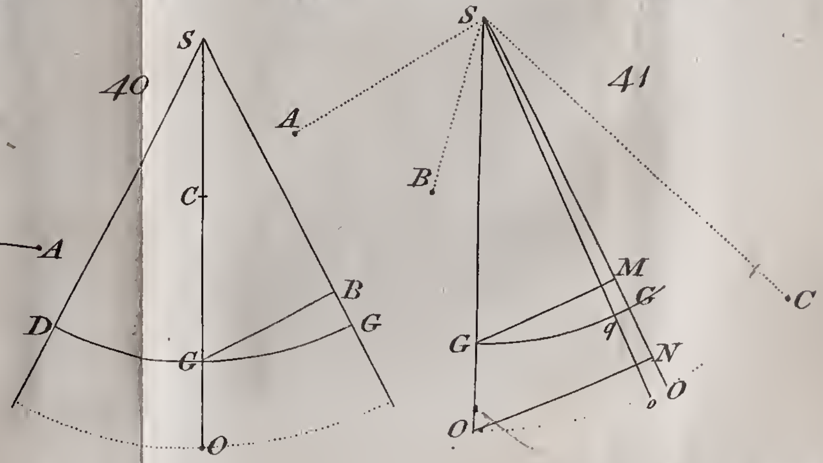
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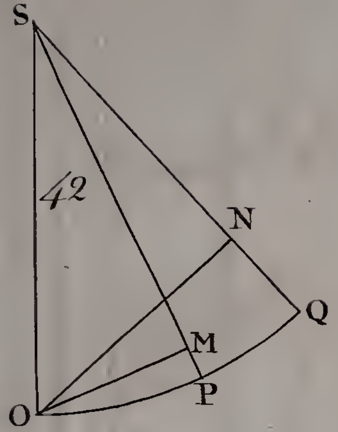

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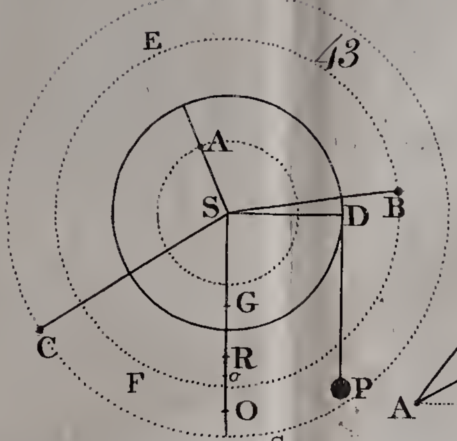

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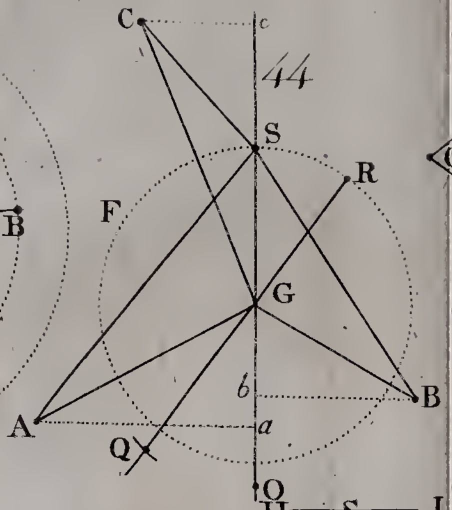

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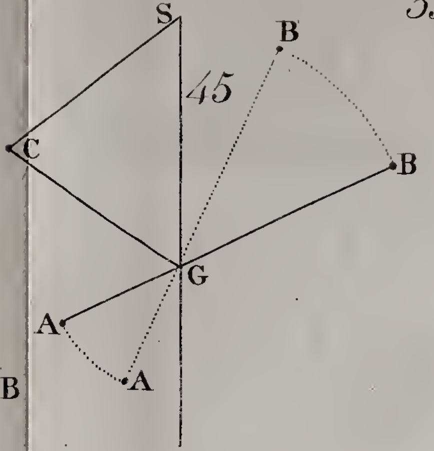

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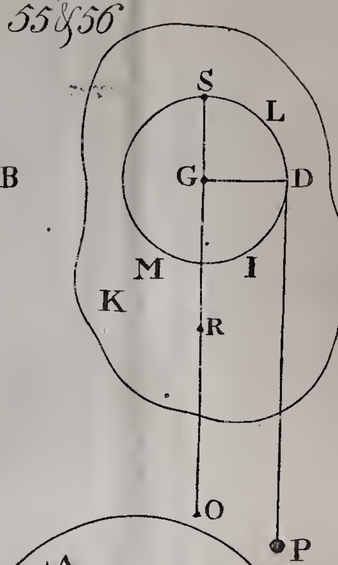

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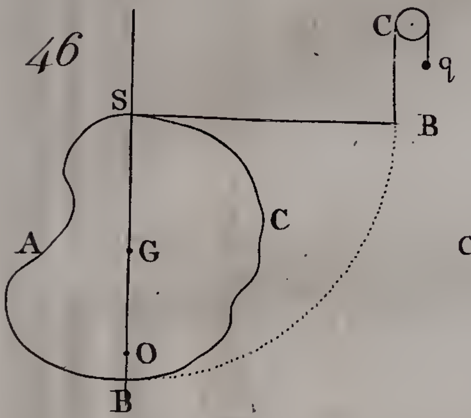

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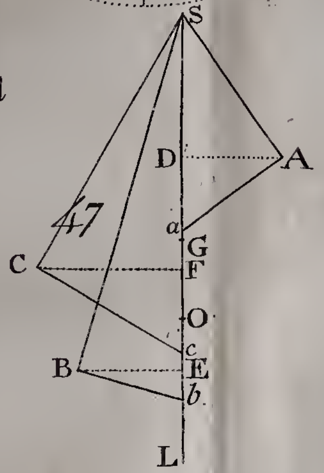

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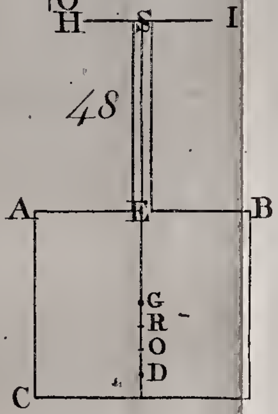

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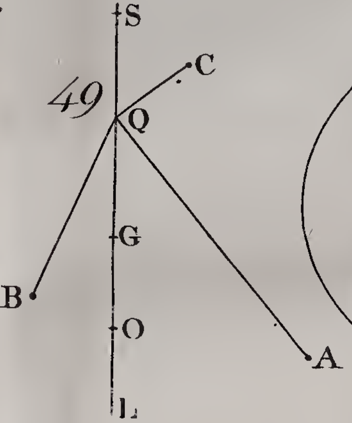

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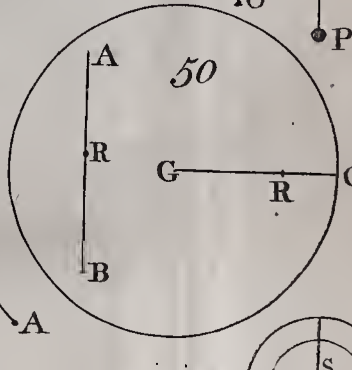

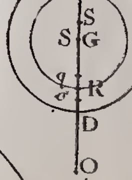

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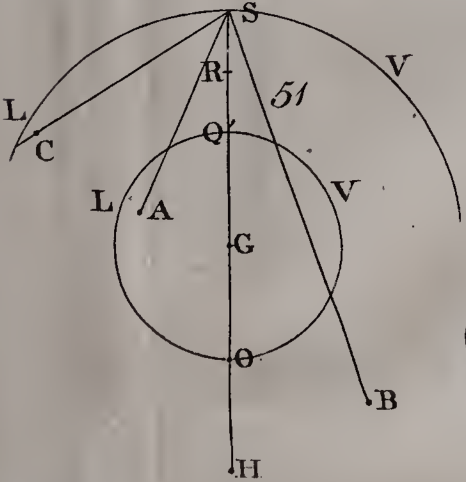

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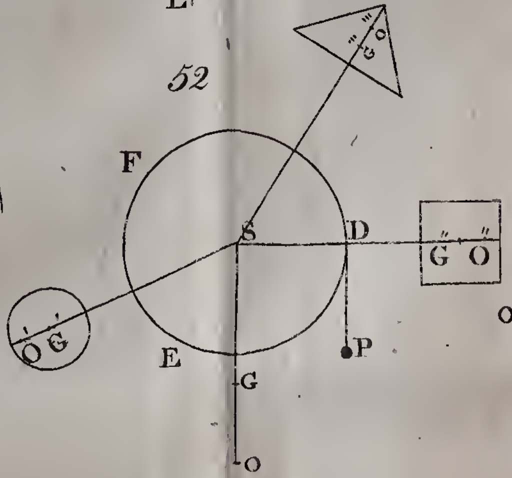

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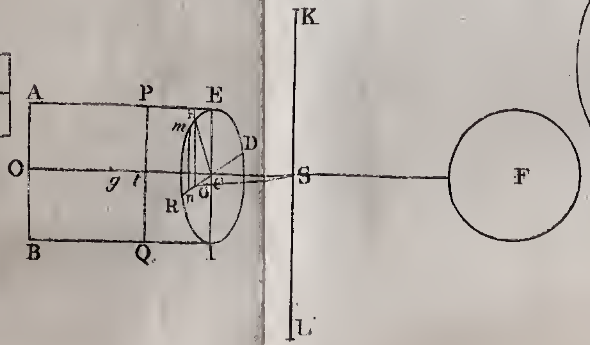

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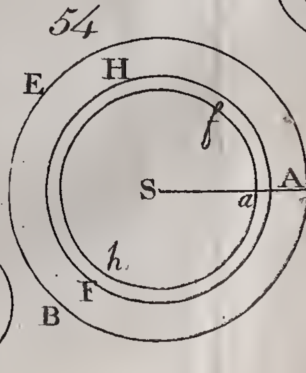

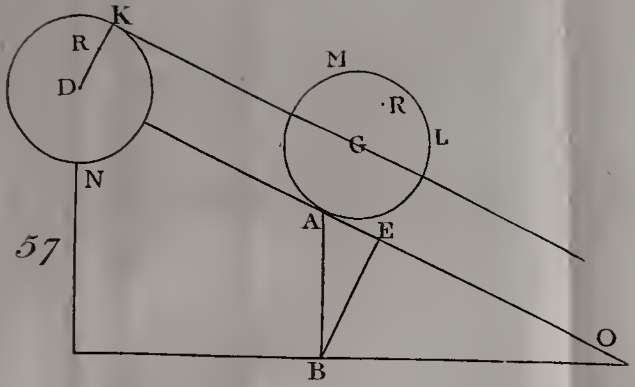

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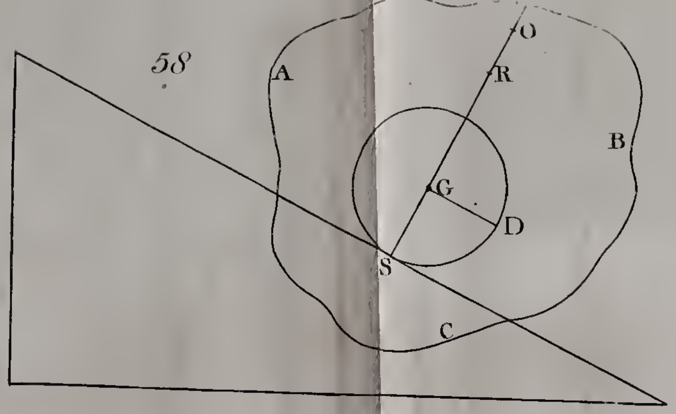

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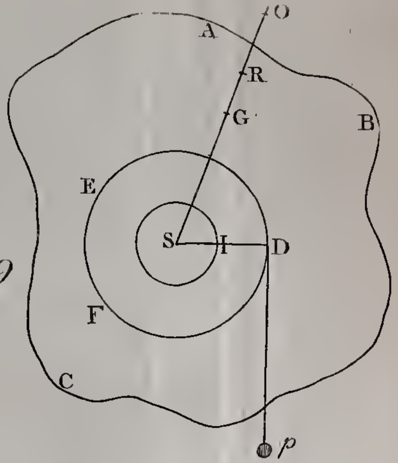

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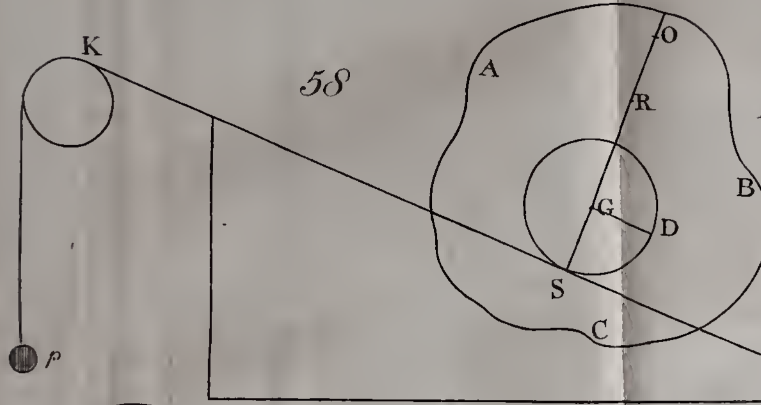

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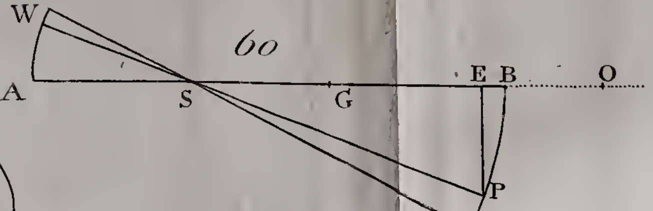

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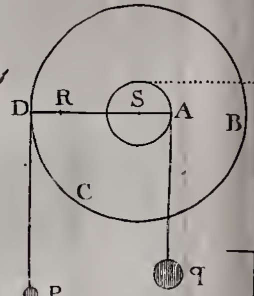

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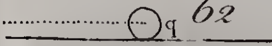

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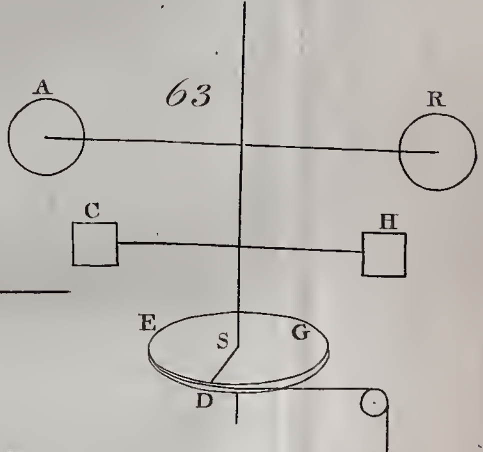

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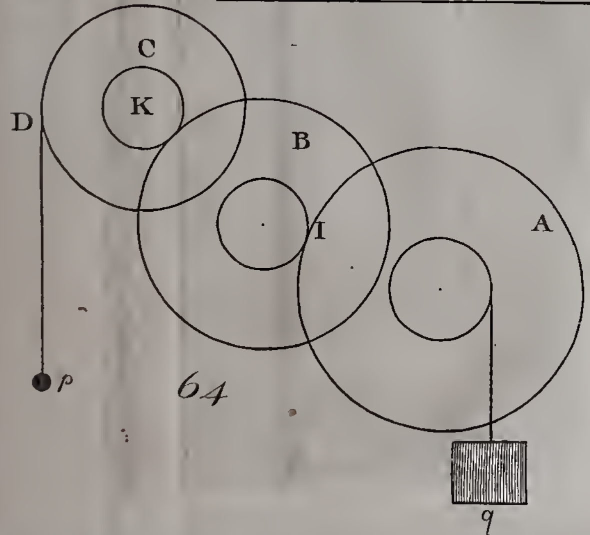

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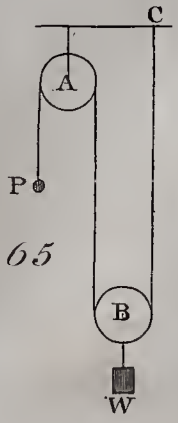

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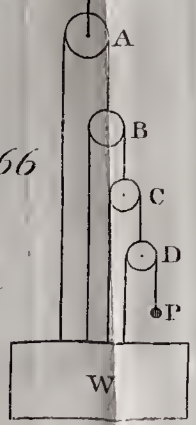

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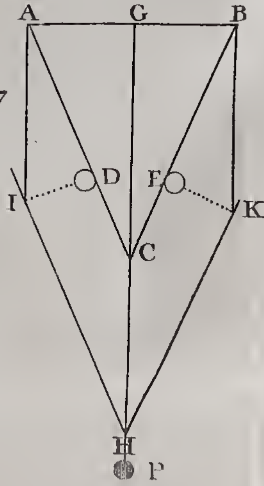

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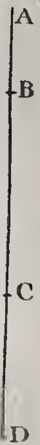

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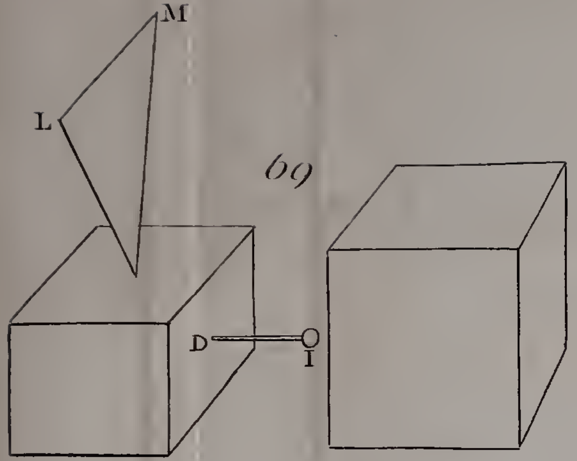

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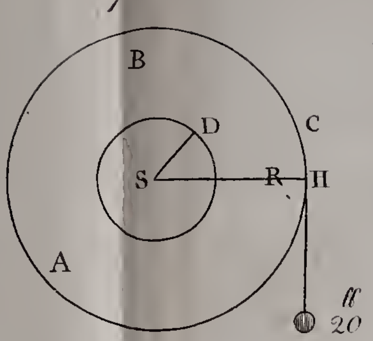

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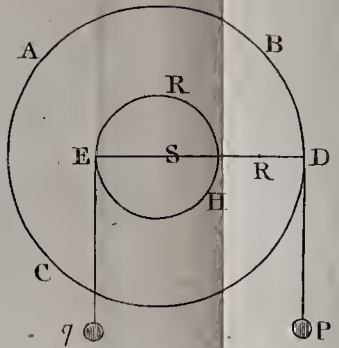

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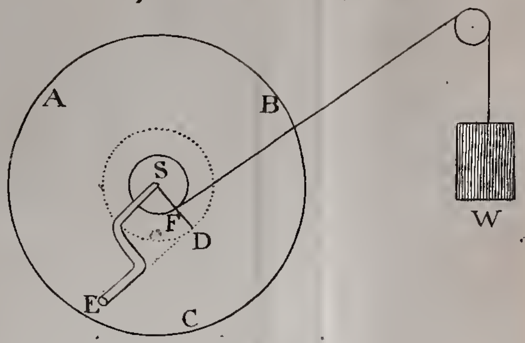

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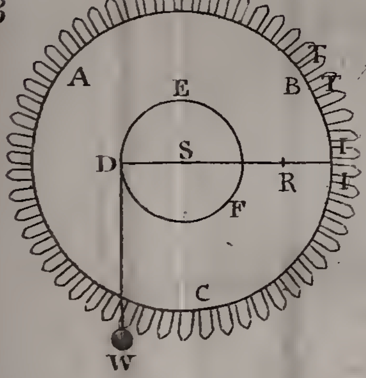

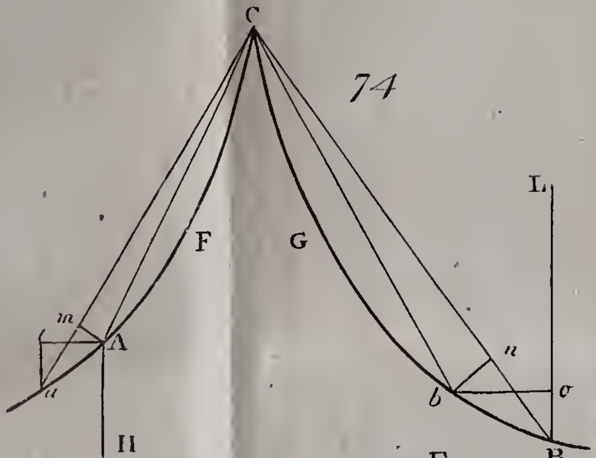

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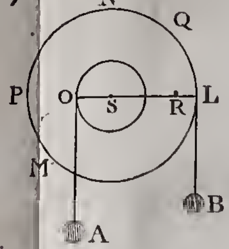

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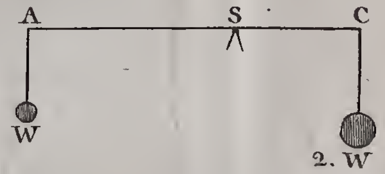

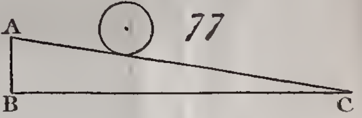

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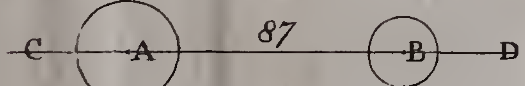

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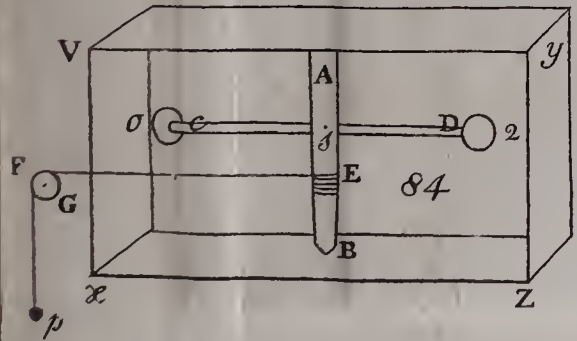

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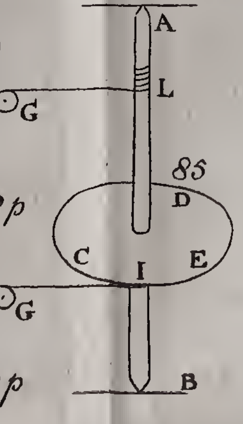

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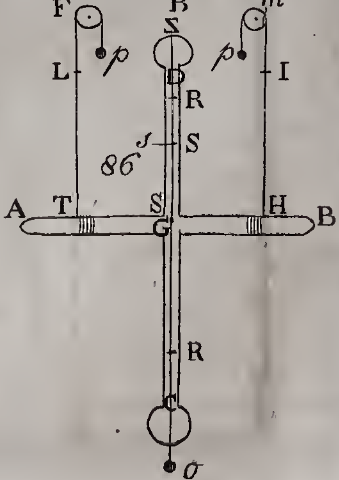

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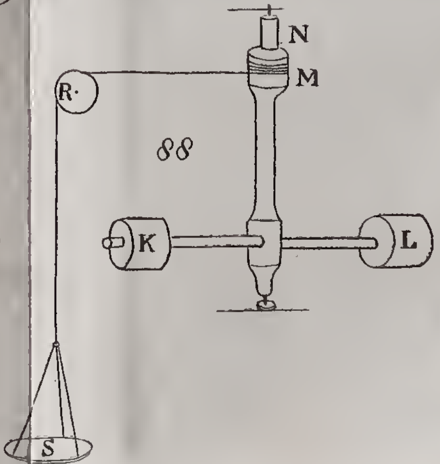

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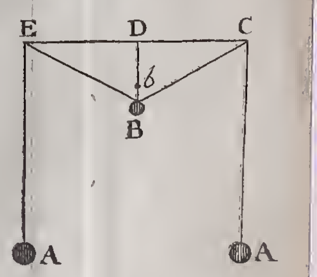

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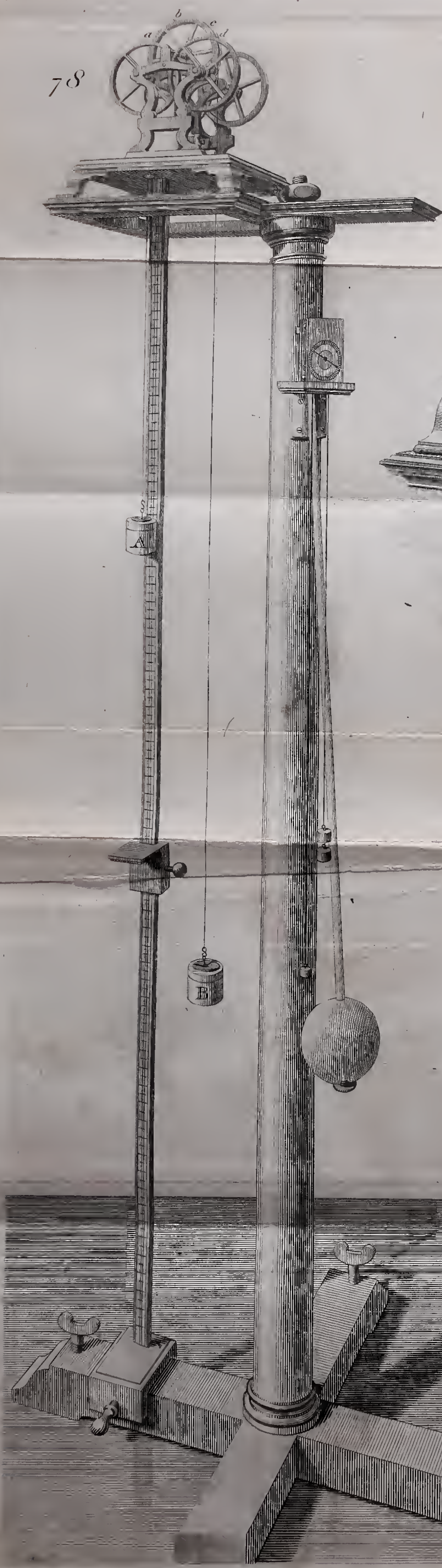

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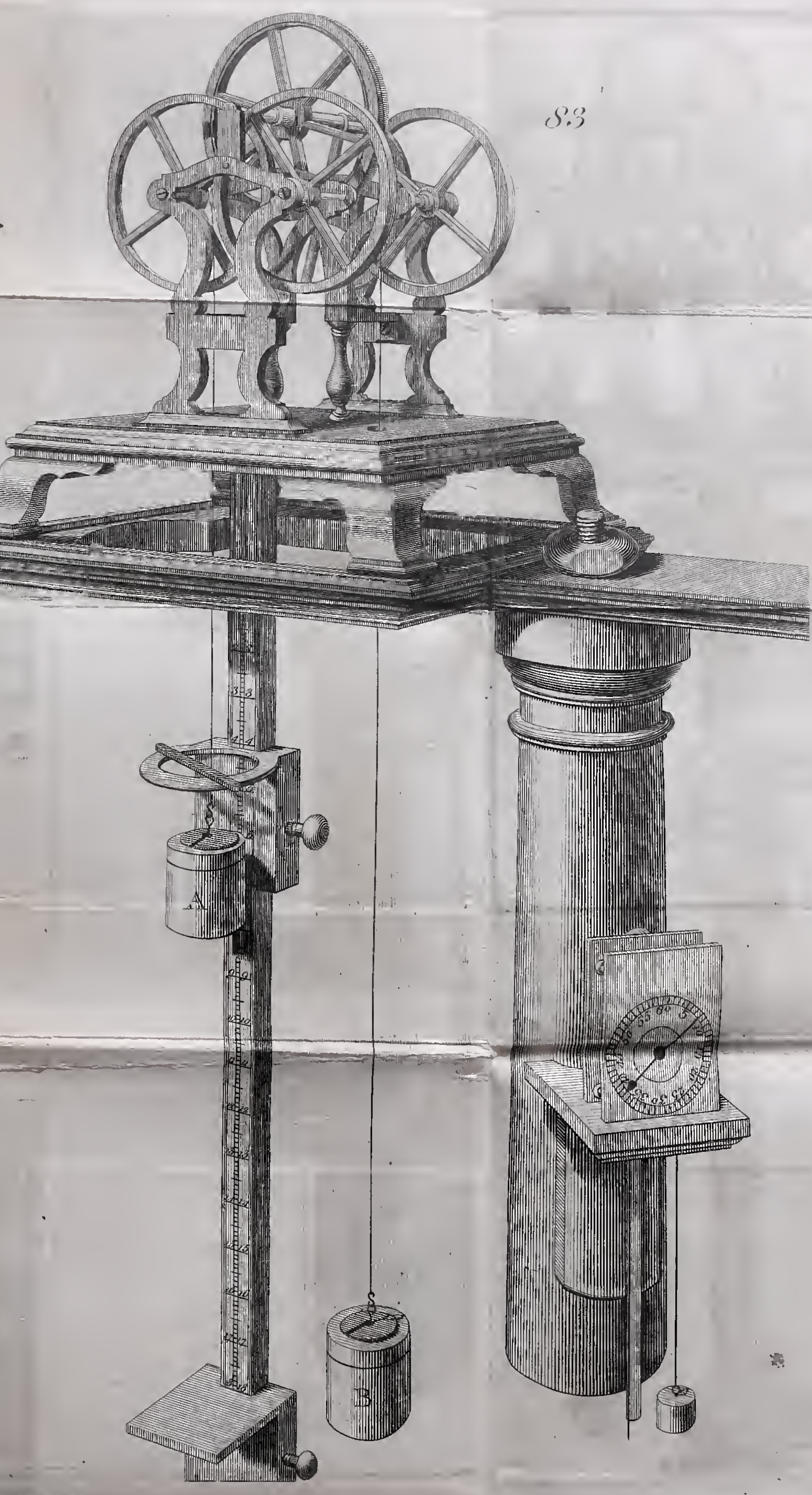

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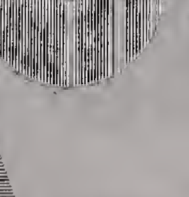

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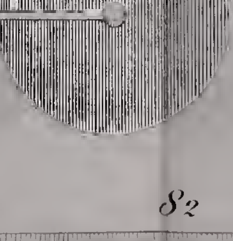

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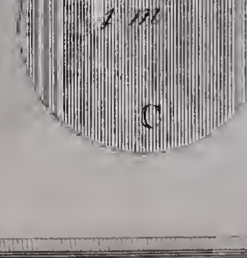

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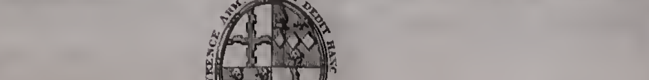

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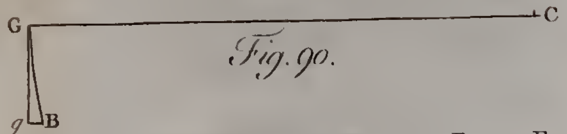

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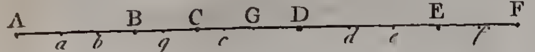

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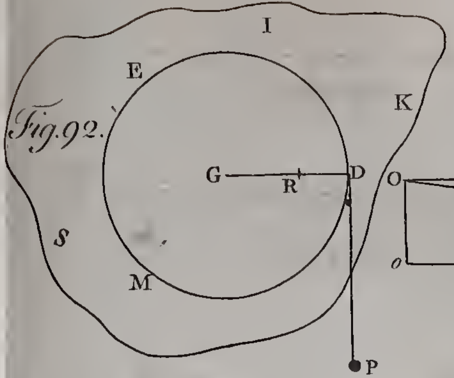

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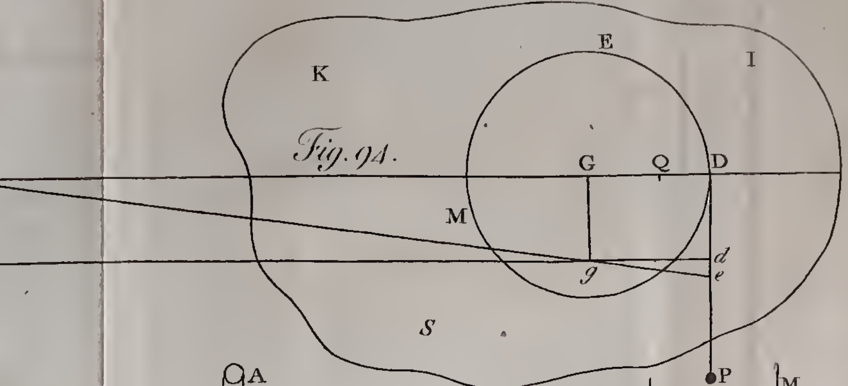

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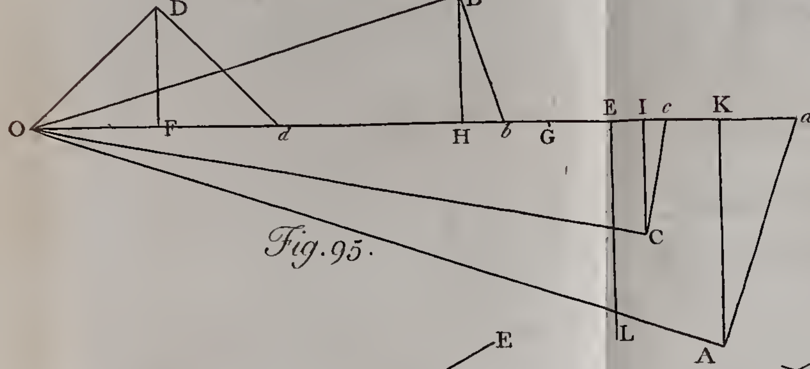

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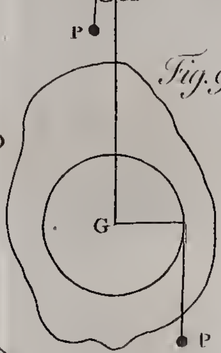

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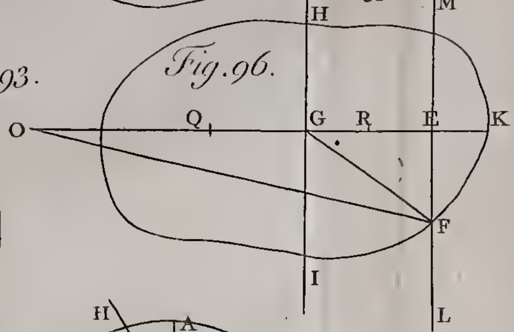

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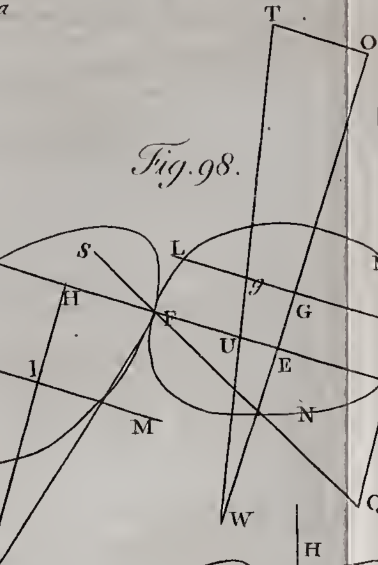

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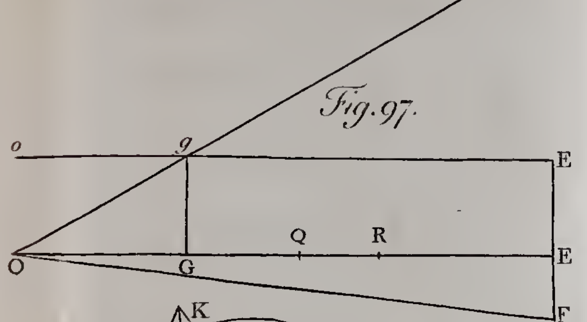

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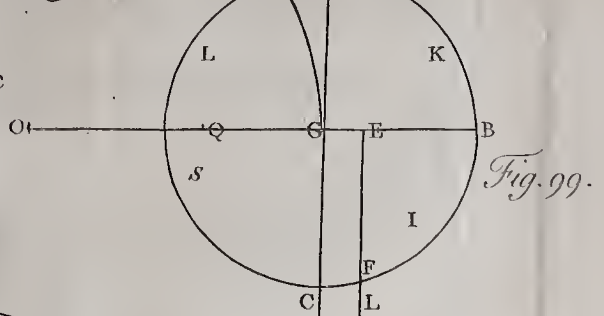

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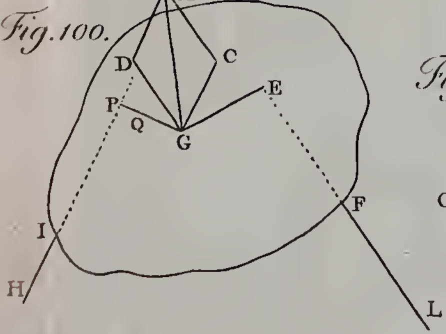

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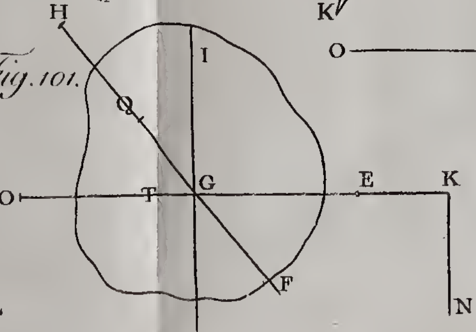

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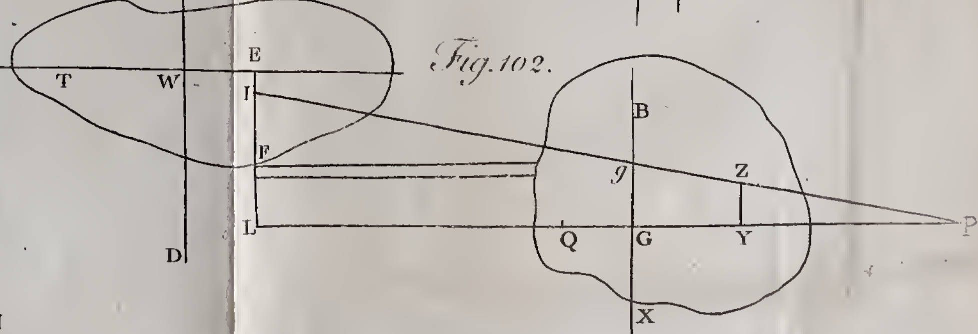

Fig. 102.

ASIRE Sc.